

Blocking Combinatorial Games

by Arthur Holshouser and Harold Reiter

Arthur Holshouser

3600 Bullard St.

Charlotte, NC,

USA, 28208

Harold Reiter

Department of Mathematics

UNC Charlotte

Charlotte, NC 28223

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We are given an acyclic directed graph (V, E) with vertex set V and directed edge set E . Acyclic means that the graph has no directed cycles. A “regular” combinatorial game is a two player game in which the positions are the vertices of such a graph and the moves correspond to the directed edges of the graph. Players alternate moving. Let T be the set of terminal positions, which we assume is not empty. A vertex is *terminal* if there are no directed edges leaving it, that is, its out degree is zero. Also, we assume that for every non-terminal vertex P , there is a positive integer $k(P)$ such that no matter how the players play, they will land on a terminal position in no more than $k(P)$ moves. The winner of the game is the player who first moves to a terminal position. In the “blocking” version of this game, in addition to these specifications, we also have the following.

First, if there is a directed edge from vertex P to vertex Q , we call Q a *successor* of P . Let $R(P)$ denote the set of successors of P . For every non-terminal vertex P , we are given a family of subsets of $R(P)$, possibly including the empty subset, which we denote by $F(P)$. Thus, $F(P)$ is a collection of sets, possibly including the empty set, and $\forall X \in F(P), X \subseteq R(P)$. We also agree that for all non-terminal vertices P , $F(P)$ is not empty. However, it is possible that for some P , $F(P) = \{\phi\}$. We denote this entire structure by (V, E, F) .

As in the regular game, the two players alternate moving. However, on each turn (including the first turn), before the moving player moves, the opposing player

must block some of his moves in the following way. Suppose the moving player is on vertex P . Then before he moves, the opposing player first chooses an $X \in F(P)$. The moving player can then move to any successor of P not in X . If $X = R(P)$, the moving player cannot move at all.

The winner is the player who makes the last move in the game. This means the winner is the one who moves onto a terminal position or else moves to a non-terminal vertex P such that he can completely block the next moving player from moving at all.

Note that a non-blocking combinatorial game (V, E) is equivalent to the blocking game (V, E, F) , where $F(P) = \{\phi\}$ for each non-terminal vertex P .

Nim values of the Blocking Combinatorial Game: We assign a non-negative integer $g(P)$ to each vertex P as follows. First, all terminal positions are assigned the value of 0. If all the successors of a position P have been assigned, then $g(P)$ is computed as follows.

Suppose that M is the unique non-negative integer such that

1. $\forall n \in \{0, 1, 2, 3, \dots, M-1\}, \forall X \in F(P), \quad [\exists Q \in R(P) \setminus X \text{ such that } g(Q) = n]$

and

2. $\exists X \in F(P)$ such that $[\forall Q \in R(P) \setminus X, \quad g(Q) \neq M]$.

Then we define $g(P) = M$. Note that $\{0, 1, 2, \dots, M-1\}$ is an empty set when $M = 0$.

In other words, $\forall n \in \{0, 1, 2, 3, \dots, M - 1\}$, the moving player can always (no matter how the blocking player blocks) move to a vertex Q such that $g(Q) = n$. Also, it is possible for the blocking player to block the moving player in such a way that the moving player cannot move to any vertex Q having $g(Q) = M$. Then we define $g(P)$ to be the number M .

Another way to state this is as follows. Define $G_P = \{n : \forall X \in F(P), \exists Q \in R(P) \setminus X \text{ such that } g(Q) = n\}$. Then $g(P) = \text{mex } G_P$, where mex is defined as follows. Suppose S is any finite set of non-negative integers. Then $\text{mex } S$ is the smallest non-negative integer that is not in S . Thus $\text{mex}\{0, 1, 2, 5, 10\} = 3$ since 3 is the smallest non-negative integer that is not in S . Also, $\text{mex}\{1, 5, 6\} = 0$.

Composite Blocking Combinatorial Games:

Suppose we start with a finite collection of blocking combinatorial games $G_1 = (V_1, E_1, F_1), G_2 = (V_2, E_2, F_2), \dots, G_n = (V_n, E_n, F_n)$. The games $G_i, i = 1, 2, \dots, n$ are called *component* games. We define the composite blocking combinatorial game, denoted by $G_1 \oplus G_2 \oplus \dots \oplus G_n = (V_1, E_1, F_1) \oplus (V_2, E_2, F_2) \oplus \dots \oplus (V_n, E_n, F_n)$, as follows. We imagine that a marker is placed on one vertex in each component game. As always, the two players take turns moving. On each turn, before the moving player moves, the blocking player first blocks some of his moves in the following way. Suppose the markers are on vertices P_1, P_2, \dots, P_n in games G_1, G_2, \dots, G_n respectively. The blocking player then chooses $X_1 \in F_1(P_1), X_2 \in F_2(P_2), X_3 \in F_3(P_3), \dots, X_n \in F_n(P_n)$. The moving player then chooses a component game G_i ,

in which he is able to move, and makes a move in that game. In other words, he moves the marker in G_i from vertex P_i to any vertex in the set $R_i(P_i) \setminus X_i$.

We could also, allow the moving player to first pick a game $G_i \in \{G_1, G_2, \dots, G_n\}$. Then the blocking player only places blocks in this one game G_i , after which the moving player moves in this G_i . The drawback with this definition is that after the blocking player blocks in G_i , it may not be possible for the moving player to move at all in G_i . At this point the moving player would have to choose a different $G_j \neq G_i$ etc. As always, the game ends when one of the two players cannot move and the winner is the player who makes the last move in the game.

Notation: In the composite blocking game $(V_1, E_1, F_1) \oplus (V_2, E_2, F_2) \oplus \dots \oplus (V_n, E_n, F_n)$, we denote the set of positions as $\{(P_1, P_2, \dots, P_n) : P_1 \in V_1, P_2 \in V_2, \dots, P_n \in V_n\}$. If the position before the moving player moves is (P_1, P_2, \dots, P_n) , then the position after he moves is $(P_1, P_2, \dots, P_{i-1}, \bar{P}_i, P_{i+1}, \dots, P_n)$, where \bar{P}_i is a successor of P_i in game (V_i, E_i, F_i) .

Problem: Suppose we know the nim values $g(P_1), P_1 \in V_1, g(P_2), P_2 \in V_2, \dots, g(P_n), P_n \in V_n$. We would like to compute $g(P_1, P_2, \dots, P_n)$, the nim value of (P_1, P_2, \dots, P_n) in $(V_1, E_1, F_1) \oplus \dots \oplus (V_n, E_n, F_n)$.

To solve this problem, let us review what we know about the non-blocking version. In the non-blocking composite combinatorial game $(V_1, E_1) \oplus \dots \oplus (V_n, E_n)$, $g(P_1, P_2, \dots, P_n) = g(P_1) \oplus g(P_2) \oplus \dots \oplus g(P_n)$ where \oplus is defined below.

Definitions: \oplus is the Abelian group operator defined on the non-negative integers

as follows. If U, V are non-negative integers, we first write U and V in binary and add the two binary numbers digitwise using the following digit table:

\oplus	0	1
0	0	1
1	1	0

As an example,

$$\begin{array}{r}
 1001101110 \\
 \oplus 101011101 \\
 \hline
 1100110011.
 \end{array}$$

Notation: If S is a finite set of non-negative integers, let us denote $mex S = S'$.

Also, if $S = \{a_1, a_2, \dots, a_k\}$ is a finite set of non-negative integers and X is a non-negative integer, as always, we define $X \oplus S = \{X \oplus a_1, X \oplus a_2, \dots, X \oplus a_k, \}$.

The following lemma follows from the review above.

Lemma. Let $S_1, S_2, S_3, \dots, S_n$, be finite sets of non-negative integers. Then

$$S'_1 \oplus S'_2 \oplus S'_3 \oplus \dots \oplus S'_n = \left[\begin{array}{l} [(S'_2 \oplus S'_3 \oplus \dots \oplus S'_n) \oplus S_1] \cup \\ [(S'_1 \oplus S'_3 \oplus S'_4 \oplus \dots \oplus S'_n) \oplus S_2] \cup \\ [(S'_1 \oplus S'_2 \oplus S'_4 \oplus \dots \oplus S'_n) \oplus S_3] \cup \dots \\ \cup [(S'_1 \oplus S'_2 \oplus \dots \oplus S'_{n-1}) \oplus S_n] \end{array} \right]'.$$

Main Theorem: Let $(V_1, E_1, F_1) \oplus (V_2, E_2, F_2) \oplus \dots \oplus (V_n, E_n, F_n)$ be a composite blocking combinatorial game. Then for all positions (P_1, P_2, \dots, P_n) ,
 $g(P_1, P_2, \dots, P_n) = g(P_n) \oplus g(P_2) \oplus \dots \oplus g(P_1)$.

Proof: Let (P_1, P_2, \dots, P_n) be a position. Define $k(P_1, P_2, \dots, P_n)$ to be the greatest possible number of moves that the two players can make before they reach a terminal vertex in the composite graph. Of course, (P_1, P_2, \dots, P_n) is a terminal vertex in the composite graph if and only if $\forall i, P_i$ is a terminal vertex in (V_i, E_i, F_i) . Also, if (P_1, P_2, \dots, P_n) is a terminal vertex, $k(P_1, P_2, \dots, P_n) = 0$.

We will prove the theorem by mathematical induction on $k(P_1, P_2, \dots, P_n)$. Obviously, the result is true when $k(P_1, P_2, \dots, P_n) = 0$.

Now $\forall (V_i, E_i, F_i), \forall P_i \in V_i, \forall x \in \{0, 1, 2, 3, \dots, g(P_i) - 1\}$, we know that the moving player can always move a marker at P_i to a new position \overline{P}_i such that $g(\overline{P}_i) = x$. Of course, if he makes the move $(P_1, P_2, \dots, P_{i-1}, P_i, P_{i+1}, \dots, P_n) \mapsto (P_1, P_2, \dots, P_{i-1}, \overline{P}_i, P_{i+1}, \dots, P_n)$, then by induction the nim value of this new position is $g(P_1, P_2, \dots, P_{i-1}, \overline{P}_i, P_{i+1}, \dots, P_n) = g(P_1) \oplus g(P_2) \oplus \dots \oplus g(P_{i-1}) \oplus g(\overline{P}_i) \oplus g(P_{i+1}) \oplus \dots \oplus g(P_n)$.

Therefore, no matter how the blocking player blocks, the moving player can always move to any nim value that is in the following set:

$$\begin{aligned}
 & [(g(P_2) \oplus g(P_3) \oplus \dots \oplus g(P_n)) \oplus \{0, 1, 2, 3, \dots, g(P_1) - 1\}] \cup \\
 (***) = & [(g(P_1) \oplus g(P_3) \oplus \dots \oplus g(P_n)) \oplus \{0, 1, 2, 3, \dots, g(P_2) - 1\}] \cup \\
 & [(g(P_1) \oplus g(P_2) \oplus g(P_4) \oplus \dots \oplus g(P_n)) \oplus \{0, 1, 2, 3, \dots, g(P_3) - 1\}] \cup \dots \\
 & [(g(P_1) \oplus g(P_2) \oplus \dots \oplus g(P_{n-1})) \oplus \{0, 1, 2, 3, \dots, g(P_n) - 1\}].
 \end{aligned}$$

However, $\forall (V_i, E_i, F_i), \forall P_i \in V_i$, it is always possible for the blocking player to block

the moving player in such a way that the moving player cannot move a marker at P_i to a new position \overline{P}_i whose nim value is $g(\overline{P}_i) = g(P_i)$. Since \oplus is an abelian group operator, we know that $a_1 \oplus a_2 \oplus \cdots \oplus a_k \oplus x = a_1 \oplus a_2 \oplus \cdots \oplus a_k \oplus y$ if and only if $x = y$. Using this fact we see that by induction on k (since the moving player only moves a marker in one game), that the blocking player can prevent the moving player from moving from (P_1, P_2, \dots, P_n) to a new position $(P_1, P_2, \dots, P_{i-1}, \overline{P}_i, P_{i+1}, \dots, P_n)$ such that $g(P_1, P_2, \dots, P_{i-1}, \overline{P}_i, P_{i+1}, \dots, P_n) = g(P_1) \oplus g(P_2) \oplus \cdots \oplus g(P_{i-1}) \oplus g(\overline{P}_i) \oplus g(P_{i+1}) \oplus \cdots \oplus g(P_n) = g(P_1) \oplus g(P_2) \oplus \cdots \oplus g(P_{i-1}) \oplus g(P_i) \oplus g(P_{i+1}) \oplus \cdots \oplus g(P_n)$. This is because $g(P_i) \neq g(\overline{P}_i)$.

Since $mex(***) = g(P_1) \oplus g(P_2) \oplus \cdots \oplus g(P_n)$, (see below for an explanation of this), then from the definition of the nim value, we see that $g(P_1, P_2, \dots, P_n) = g(P_1) \oplus g(P_2) \oplus \cdots \oplus g(P_n)$.

If it is not clear to the reader why $mex(***) = g(P_1) \oplus g(P_2) \oplus \cdots \oplus g(P_n)$, first define $S_1 = \{0, 1, 2, \dots, g(P_1) - 1\}$, $S_2 = \{0, 1, 2, \dots, g(P_2) - 1\}$, $S_3 = \{0, 1, 2, \dots, g(P_3) - 1\}$, \dots , $S_n = \{0, 1, 2, \dots, g(P_n) - 1\}$. Now $mex S_1 = S'_1 = g(P_1)$, $mex S_2 = S'_2 = g(P_2)$, \dots , $mex S_n = S'_n = g(P_n)$. Substituting this in the formula specified earlier, we see that $mex(***) = g(P_1) \oplus g(P_2) \oplus \cdots \oplus g(P_n)$. ■

Strategy for Playing a Composite Blocking Combinatorial Game:

The strategy for playing a composite blocking combinatorial game is very similar to the strategy for the non-blocking version.

First, the nim values are computed for each vertex of $(V_1, E_1, F_1), (V_2, E_2, F_2)$,

$\cdots (V_n, E_n, F_n)$. The nim value of any position (P_1, P_2, \cdots, P_n) in the composite game is computed by $g(P_1, P_2, \cdots, P_n) = g(P_1) \oplus g(P_2) \oplus \cdots \oplus g(P_n)$. As in the regular version, we say that a position (P_1, P_2, \cdots, P_n) is *balanced* if $g(P_1, P_2, \cdots, P_n) = 0$. Of course, all terminal positions (P_1, P_2, \cdots, P_n) are balanced since $g(P_1, P_2, \cdots, P_n) = g(P_1) \oplus g(P_2) \oplus \cdots \oplus g(P_n) = 0 \oplus 0 \oplus \cdots \oplus 0 = 0$. We now show that all balanced positions (P_1, P_2, \cdots, P_n) are safe positions and all unbalanced positions are unsafe positions. First, suppose (P_1, P_2, \cdots, P_n) is balanced. This means $g(P_1) \oplus g(P_2) \oplus \cdots \oplus g(P_n) = 0$. Now by the definition of the nim values in each component game, (V_i, E_i, F_i) , the blocking player can always force the moving player to move from $P_i \mapsto \bar{P}_i$ such that $g(\bar{P}_i) \neq g(P_i)$. Therefore, no matter what component (V_i, E_i, F_i) the moving player chooses, the blocking player can force the moving player to move to a different nim value in that component. Since \oplus is an Abelian group operator, this means the nim value of the new position $(P_1, P_2, \cdots, P_{i-1}, \bar{P}_i, P_{i+1}, \cdots, P_n)$ cannot equal 0. Therefore, the moving player is forced to move to an unbalanced position.

Next, suppose (P_1, P_2, \cdots, P_n) is unbalanced. This means $g(P_1, P_2, \cdots, P_n) = g(P_1) \oplus g(P_2) \oplus \cdots \oplus g(P_n) \neq 0$.

Now by the definition of the nim values in each component game, (V_i, E_i, F_i) , we know that no matter how the blocking player blocks, the moving player can always move from $P_i \mapsto \bar{P}_i$ in such a way that $g(\bar{P}_i)$ can be any member of the set $\{0, 1, 2, 3, \cdots, g(P_i) - 1\}$. Therefore, as in the regular version (and also Bouton's

Nim), the moving player can always choose a component (V_i, E_i, F_i) and move $P_i \mapsto \overline{P}_i$ in that component so that $g(P_1, P_2, \dots, P_{i-1}, \overline{P}_i, P_{i+1}, \dots, P_n) = g(P_1) \oplus g(P_2) \oplus \dots \oplus g(P_{i-1}) \oplus g(\overline{P}_i) \oplus g(P_{i+1}) \oplus \dots \oplus g(P_n) = 0$. That is, the moving player can always move from an unbalanced position to a balanced position. Since all terminal positions are balanced, this means that the balancing player must always be the winner. Therefore, the balanced positions are the safe positions, and the unbalanced positions are the unsafe positions.

A Special Type of Blocking Combinatorial Game:

There is a special type of blocking combinatorial game that deserves to be discussed separately. Suppose we are given the acyclic directed graph (V, E) of a regular combinatorial game. Also, in advance of the game, the two players fix a positive integer K . Thus the entire structure is denoted by (V, E, K) .

As always, the two players alternate moving. However, on each turn (including the first turn) before the moving player moves, the opposing player places a block on at most K of the moving player's moves. He can do this in any way that he chooses, including the option of blocking less than K moves (or blocking no moves at all for that matter).

As always, the winner is the one who makes the last move in the game. This means that the winner is the one who moves onto a terminal position or else moves to a non-terminal position whose number of successors does not exceed K .

As always, we assign a non-negative integer $g(P)$ to each vertex P , called the

nim value. All the terminal positions are assigned the value of 0. If all the successors of a position P have been assigned, then $g(P) = \text{mex}_k\{g(Q) : Q \text{ is a successor of } P\}$, where mex_k is defined below and where the set is a multiset.

Definition: Suppose S is a finite multiset of non-negative integers. A multiset is a set in which elements are allowed multiple membership. For example, $S = \{0, 0, 1, 2, 3, 3, 3, 5, 5, 5, 7, 8, 8, 10, 10\}$. Then $\text{mex}_k S$ is defined as the smallest non-negative integer M such that the number of copies of M in S does not exceed K .

Thus, each $n \in \{0, 1, 2, 3, \dots, M - 1\}$ appears at least $K + 1$ times in S and M appears no more than K times in S . We could denote the ordinary $\text{mex } S = \text{mex}_0 S$.

Theorem: Suppose we are given the acyclic directed graph (V, E) of a regular combinatorial game. Suppose the nim values have been computed for the vertices of (V, E) . Suppose these nim values satisfy the following “special” condition. For any vertex N , if $X \neq Y$ are any two distinct successors of N , then $g(X) \neq g(Y)$.

Choose any $K \in \mathbb{Z}^+$ and play the blocking game (V, E, K) that was just specified. That is, the opposing player on each turn (including the first turn) can block up to K of the moving player’s moves.

Let $g_K(N), N \in V$, denote the nim values of the blocking game. Then $\forall N \in V, g_K(N) = \lfloor \frac{g(N)}{K+1} \rfloor$, where $\lfloor X \rfloor$ is the floor function. The proof is reasonably easy and is left to the reader.

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