A Nim-like Game on the Integers

The purpose of this paper is to investigate a new nim-like combinatorial game. We start with an ordered list $X = x_1 \leq x_2 \leq \ldots \leq x_N$ of non-negative integers. Repetition of values is allowed. Such a list is called a *position.* Two players alternate choosing a number x_i in the position and replacing it by the elements of any set $\{u_1, u_2, \ldots, u_k\}$ of non-negative integers whose largest element is less than x_i. For example, $(0, 1, 3, 3, 6, 7, 9) \longmapsto (0, 1, 1, 2, 3, 3, 4, 7, 9)$ represents the move in which $x_5 = 6$ is replaced by the set $\{1, 2, 4\}$. If a player chooses the number 0 from the position, then the resulting position is just the original position with the 0 removed. Of course the new position is obtained by reordering the new set of integers.

We are given as well two collections $\mathcal{L} = \{L_1, L_2, \ldots, L_k\}$ and $\mathcal{W} = \{W_1, W_2, \ldots, W_l\}$ of non-emplty sets of non-negative integers, called respectively, the *losing sets* and the *winning sets.* For each i, let l_i denote the smallest element of L_i. When little possibility of confusion exists, we will talk about the <u>set</u> X of members of the position X. For example, the position $X = (1, 1, 1, 2, 3, 3, 4, 7, 9)$ will sometimes be referred to as $\{1, 2, 3, 4, 7, 9\}$.

We assume that initially $X \cap L_i \neq \phi$ and $X \cap W_j \neq \phi$ for all i, j. Let $\mathcal{T} = \mathcal{L} \cup \mathcal{W}$. We call $\mathcal{T}$ the terminal sets, and any position X satisfying $X \cap T = \phi$, where $T \in \mathcal{T}$, is called a *terminal position.* Now let $\mathcal{T}$ be ordered in any way subject to the condition that if

$$l_i \in W_j, \text{ then } L_i \text{ is listed before } W_j. \qquad **$$

The game ends as soon as a position X is reached such that $X \cap T = \phi$, where $T \in \mathcal{T}$. Let T' denote the first terminal set of the ordered list $\mathcal{T}$ whose intersection with X is empty. The player who moves to such a terminal position is the winner or loser depending on whether T' belongs to $\mathcal{W}$ or $\mathcal{L}$, and the final position of the game is called a *winning position* or a *losing position*, respectively.

<u>Strategy</u>. Let θ be a non-negative integer and let $\mathcal{L}_\theta = \{L_i \in \mathcal{L} |\, l_i = \theta\}$. That is, $\mathcal{L}_\theta$ is the collection of L_i whose smallest element is θ. We call a position $X = x_1 \leq x_2 \leq \ldots \leq x_N$ *θ-balanced* if the following conditions are satisfied.

1. If $\mathcal{L}_\theta = \phi$, then θ appears an even number of times in X.
2. If some member of $\mathcal{L}_\theta$ is a singleton set, then θ appears an odd number of times in X.
3. If $\mathcal{L}_\theta$ is neither empty nor contains any singletons, then let $\mathcal{L}'_\theta$ denote the family of all members of $\mathcal{L}_\theta$ with θ removed from each. That is, if $\mathcal{L}_\theta = \{L_{i_1}, L_{i_2}, \ldots, L_{i_t}\}$, then $\mathcal{L}'_\theta = \{L_{i_1} \setminus \{\theta\}, L_{i_2} \setminus \{\theta\}, \ldots, L_{i_t} \setminus \{\theta\}\}$.
   (a) If some member L of $\mathcal{L}'_\theta$ is disjoint from X, then the number of times θ appears in X is odd, and

(b) if, on the other hand, each member of $\mathcal{L}'_\theta$ has non-empty intersection with X, then θ appears in X an even number of times.

Finally, we say a position X is balanced if X is θ-balanced for all θ.

The Theorem. Suppose $(X, \mathcal{L}, \mathcal{W})$ is given. Then the second player wins if and only if X be balanced.

Proof. We prove the theorem by showing that

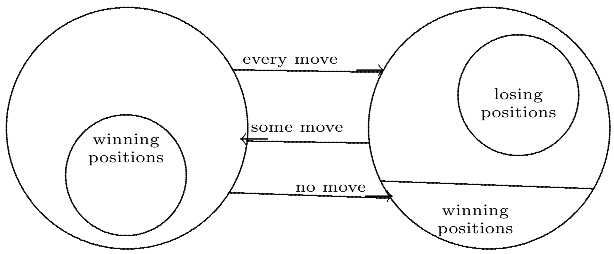
A. Every move from a balanced, non-terminal position results in an unbalanced position,

B. Every move from a balanced, non-terminal position results in a non-winning position, and

C. From any unbalanced, non-terminal position, there is a move to a balanced position.

D. No balanced position is a losing position.

The figure shows the mains ideas of the proof.



balanced positions unbalanced positions

Proof of A. Suppose X is balanced, non-terminal position, and suppose the next player chooses the number θ. He replaces θ by numbers less than θ or by nothing at all. This move is denoted by $X \longmapsto X'$. This move does not change any of the numbers larger than θ, and therefore the parity requirement for number of appearences of θ in X' for X' to be θ-balanced does not change, by the definition of θ-balanced. But the parity of the number of appearences of θ changes. Therefore X' is θ-unbalanced, and therefore unbalanced.

Proof of B. Suppose there is a move $X \longmapsto X'$ from the balanced non-terminal position X to a winning position X'. Examine the list of terminal sets $\mathcal{T}$, and note that if X' is a winning position, then $X' \cap W_j = \phi$ for some W_j and $X' \cap L_i \neq \phi$ for all L_i appearing before W_j in the list. If X' resulted from X by removing θ from the table, it was the removal of θ that created the disjointness of the two sets X' and W_j since X was not a terminal position. That means $\theta \in W_j$, because otherwise, the game would have ended earlier. Also, θ must appear only once in X since each move can remove only one occurance of a number. Since the number of occurances of θ in X is odd (one), and X is balanced, it follows from the definition of balanced that either some member of $\mathcal{L}_\theta$ is a singleton set or $\mathcal{L}_\theta$ is neither empty nor contains any singletons, and some member of $\mathcal{L}'_\theta = \{L_{i_1} \setminus \{\theta\}, L_{i_2} \setminus \{\theta\}, \ldots, L_{i_t} \setminus \{\theta\}\}$ is disjoint from X. Thus there is an L_i whose intersection with X is $\{\theta\}$ and $l_i = \theta$. But this implies that L_i must appear before W_j in $\mathcal{T}$, and therefore X' cannot be a winning position, because of condition $**$.

Proof of C. Suppose X is non-terminal and unbalanced. Let θ denote the largest number for which X is not θ-balanced. To see that θ actually appears on the table, note that if θ appears an odd number of times in X, then it appears at least once, and if it appears and even number of times, then it should appear an odd number of times. Then either condition 2. or condition 3a. holds, each of which implies that θ appears at least twice in X. It is easy to see that a balancing move $X \longmapsto X'$ can be made by removing θ from the table, and replacing it in the order $\theta - 1$, $\theta - 2$, $\ldots, 0$ by whichever of the numbers less than θ are required for X' to be balanced. In other words, for each $\alpha < \theta$, include another α in X' if the parity of α needs changing, and don't otherwise.

Proof of D. Suppose X is balanced. Then there is no L_i in $\mathcal{L}$ disjoint from X. Suppose otherwise. If $L_i \cap X = \phi$, let $\theta = l_i$. Then from 2. and 3a., θ must appear an odd number of times, a contradiction.

The game discussed here is more powerful than Bouton's Nim in the following sense. Suppose we start with the position $X = (0, 2, 4, 6)$ and restrict moves so that each move replaces a number by a single number less than it or nothing at all. Let $\mathcal{W} = \{W\}$ where $W = \{0, 1, 2, 3, 4, 5, 6\}$. In order to be disjoint from W, a position X must be empty. Now this game is equivalent to ordinary Bouton's Nim with pile sizes $1, 3, 5$, and 7.

References

[1] Richard K. Guy *Fair Game*, 2nd. ed., COMAP, New York, 1989.

[2] Berlekamp, Conway, and Guy *Winning Ways*, Academic Press, etc