

November 15, 2002

Your name \_\_\_\_\_

The first 3 problems count 7 points each and the final ones counts as marked. In the multiple choice section, circle the correct choice (or choices). You do not need to show your work on multiple choice items. You must show your work on the other problems. The total number of points available is 111.

1. The rational function  $f$  is defined by

$$f(x) = \frac{(x-4)(x-2)}{(x^2-4)(x-3)}.$$

Which of the following lines is **not** an asymptote?

- (A)  $y = 0$     (B)  $x = 3$     (C)  $x = -2$     (D)  $x = 2$     (E)  $x = 4$

**Solution:** D,E. Factor  $x^2-4 = (x-2)(x+2)$  and then remove the  $(x-2)$ 's from numerator and denominator. Thus,  $x = 2$  is not a zero of the denominator, nor is  $x = 4$ . The other three are asymptotes.

2. Which line are asymptotes of

$$f(x) = \frac{x^2(x^2-4)}{x^2(x-3)(x+2)}.$$

Circle all that apply.

- (A)  $y = 0$     (B)  $y = 1$     (C)  $x = 3$     (D)  $x = -2$     (E)  $x = 2$

**Solution:** B,C. Again you must remove the common factor  $x+2$ . Thus the asymptotes are  $x = 3$ ,  $x = 0$  and  $y = 1$  (since the degrees are the same), and the coefficients of the highest powers are both 1.

3. When  $\frac{(2x+2)(x-1)^2}{x(x-3)(x+4)}$  is expressed in rational function form

$$\frac{a_n x^n + a_{n-1} x^{n-1} + \dots + a_0}{b_m x^m + b_{m-1} x^{m-1} + \dots + b_0},$$

what is the value of  $\frac{a_n}{b_m} + a_0 + b_1$ ?

- (A) 1    (B) 5    (C) 9    (D) 11    (E) 12

**Solution:** B. For this rational function,  $n = m = 3$  and  $a_3 = 2$ ,  $b_3 = 1$ ,  $a_0 = 2$  and  $b_1 = 1$ , so the value of the expression is  $2/1 + 2 + 1 = 5$ .

4. (30 points) The rational function  $f$  is defined by

$$f(x) = \frac{(x-4)^2(3x-2)^3}{(x^2-4)(2x-7)(x+1)^2}$$

Do a complete analysis of the function discussing asymptotes of both types and intercepts. Sketch the graph on the axes provided. Notice that the numerator, but not the denominator is in factored form.

- (a) What is the domain of  $f$ ?

**Solution:** The domain is all the numbers except the zeros of the denominator,  $x = \pm 2, -1, 7/2$ .

- (b) What are the  $x$ -intercepts?

**Solution:** The zeros of the numerator are  $x = 4$  and  $x = 2/3$ .

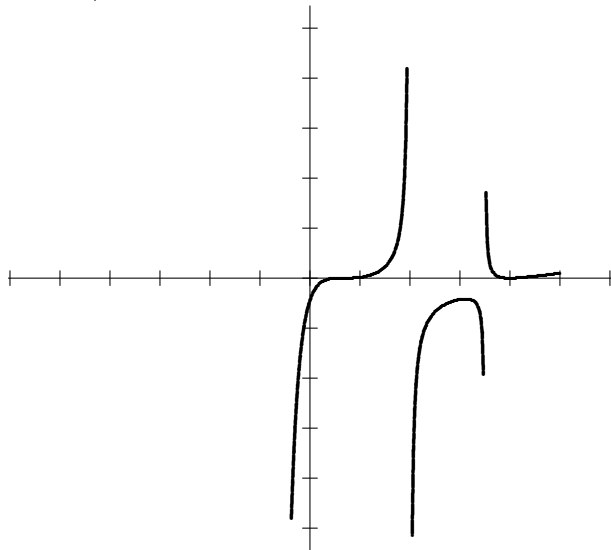
- (c) What are the vertical asymptotes?

**Solution:** These are the same as the zeros of the denominator,  $x = \pm 2, -1, 7/2$ .

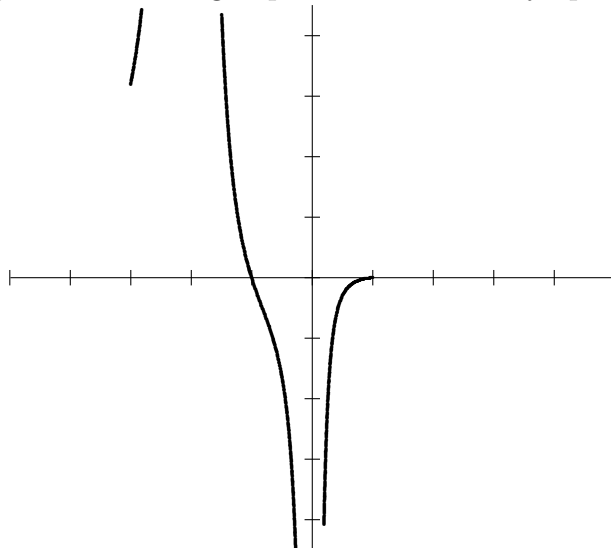
- (d) Discuss the horizontal asymptote(s)?

**Solution:** Compute the quotient of coefficients  $a_5$  and  $b_5$  to get  $27/2$ .

**Solution:** I cannot produce the complete graph. The left side of the plane contains two additional segments, one between  $-2$  and  $-1$  that looks much like the one between  $2$  and  $3.5$ , and the other has  $-2$  as an upper VA and  $y = 27/2$  as a horizontal asymptote.



5. (30 points) The graph on the grid provided below satisfies all the following. It has zeros at  $x = 1$  and  $x = -1$ , vertical asymptotes  $x = -2$  and  $x = 0$  and a horizontal asymptote  $y = 2$ . Notice that both sides of the function go down at the  $x = 0$  asymptote and both go up at the  $x = -2$  asymptote.



Find a symbolic representation of such a function.

**Solution:** The numerator must have  $x - 1$  and  $x + 1$  as factors because 1 and  $-1$  are zeros of the rational function. The denominator must have both  $(x+2)^2$  and  $x^2$  as factors because the asymptotes behave the same way on both sides of the points in question. But the function has a nonzero horizontal asymptote, so the degree of the numerator must be the same as that of the denominator. Hence we must find a way to increase the degree of the numerator without introducing any more zeros. We can do this by multiplying by either  $(x - 1)^2$  or  $(x + 1)^2$ . To make  $y = 2$  a horizontal asymptote, we can simply multiply by 2. Thus, one function that works is

$$R(x) = \frac{2(x - 1)^3(x + 1)}{x^2(x + 2)^2}$$

6. (30 points) Find all the zeros of the polynomial  $p(x) = 2x^4 - x^3 - 20x^2 + 13x + 30$ . A calculator solution to this problem is not acceptable. You must find the roots using algebra, use division to simplify the problem (ie, find the depressed polynomial), then find the zeros of the depressed polynomial repeatedly.

**Solution:** First try some of the possible rational roots,  $\pm 30, \pm 15, \pm 6, \pm 1/2$ , etc. Note that  $p(1) = 24 \neq 0$ , but  $p(-1) = 2 + 1 - 20 - 13 + 30 = 0$ , so  $x + 1$  is a factor. Division yields  $\frac{p(x)}{x+1} = 2x^3 - 3x^2 - 17x + 30 = q(x)$ . This depressed polynomial has  $x - 2$  as a factor:  $q(2) = 16 - 12 - 34 + 30 = 0$ . Long division get the depressed polynomial  $\frac{q(x)}{x-2} = (2x^2 + x - 15) = (2x - 5)(x + 3)$ . Thus the zeros are  $x = -1, 2, 5/2$ , and  $-3$ .