

Recall that two sets A and B are *equinumerous* (also called *equivalent*) if there is a bijection from A to B . We write $A \approx B$ in this case. The relation \approx is an *equivalence relation*. That is, \approx is *reflexive* ($\forall A, A \approx A$), *symmetric* ($\forall A, B, (A \approx B) \Rightarrow (B \approx A)$), and *transitive* ($\forall A, B, C, (A \approx B) \wedge (B \approx C) \Rightarrow (A \approx C)$),

1. Prove that $(0, 1] \approx (0, 1)$ by following the steps below. This is sometimes called ‘making a point disappear’.

(a) Define a function f from $(0, 1]$ to $(0, 1)$ as follows: Let

$$f(x) = \begin{cases} 1/(n+1) & \text{if } x = 1/n \text{ for some integer } n \\ x & \text{otherwise} \end{cases}$$

Thus, $f(1) = 1/2, f(1/2) = 1/3$, etc. Sketch the graph of f .

- (b) Prove that f is one-to-one. To do this, let $S = \{1, 1/2, 1/3, 1/4, \dots\}$ and consider two points a and b of $(0, 1]$. If both a and b belong to S , then \dots . If exactly one belongs to S , then \dots . Finally, if neither belongs to S , then \dots .
- (c) Prove that f is onto. Take an arbitrary point b in $(0, 1)$. Again consider whether b belongs to S . If $b \in S$, then we can find an a in $(0, 1]$ such that $f(a) = b$ as follows:
If b is not in S , then we find an a in $(0, 1]$ such that $f(a) = b$ as follows:
Complete the paragraphs started above to complete the proof.

2. Let R denote the set of real numbers. Prove that the function

$$f(x) = \begin{cases} 1/x & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

is a bijection of R to R .