December 13, 2000

Your name ______________________________

It is important that you **show your work**.

1. Use the Euclidean algorithm to solve the decanting problem for decanters of sizes 199 and 98. In other words, find integers x and y such that $199x + 98y = 1$.

2. Recall that 746 is a decimal representation; that is, a sum of multiples of powers of 10. the word 'interpret' means 'express in decimal notation'.

   (a) Interpret 10110.11_2

   (b) Interpret 110.11_{-2}

   (c) Find the base 6 representation of 2001.

   (d) Find the base 4 representation of 1/9.

3. (Modular Arithmetic) Note that each of the even powers of 10 are congruent to 1 modulo 11; ie, $10^{2n} \equiv 1 \pmod{11}$, while the odd powers of 10 are congruent to -1 modulo 11. You can use these facts to prove, for example, that $1234 \equiv -1+2-3+4 \pmod{11}$. Use the facts above together with the fact that each base-10 number is equivalent to the sum of its digits modulo 9 to

   (a) Find the remainder when 12345678910111213 is divided by 11.

   (b) Find the remainder when 234567891011121314 is divided by 9.

   (c) Find the remainder when 3456789101112131415 is divided by 99.

   (d) Find an ordered pair of digits (a, b) such that $12a6b1$ is divided by 99.

4. Let sets A, B, and C satisfy $|A| = 10, |B| = 9, |C| = c, |AB| = 6$, $|AC| = 4, |BC| = 2, |ABC| = 1$, and $|A \cup B \cup C| = 23$.

   (a) What is c?

   (b) What is $|\overline{A}\,\overline{B}\,C|$?

   (c) What is $|\overline{AB}C \cup A\overline{BC} \cup \overline{A}B\overline{C}|$?

5. Prove by mathematical induction the formula for sum of the squares of the first n positive integers:

$$1^2 + 2^2 + 3^2 + \cdot + n^2 = \frac{1}{6}n(n+1)(2n+1)$$

6. Let $\mathcal{U} = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$ be the universal set. Let $S = \{1, 2, 3, 4, 5, 6\}$ and $T = \{0, 7, 8, 9\}$. Find each of the following numbers.

   (a) How many subsets does $\mathcal{U}$ have?

   (b) How many 5-element subsets does $\mathcal{U}$ have?

   (c) How many 5-element subsets A of $\mathcal{U}$ satisfy $|A \cap S| = 3$ and $|A \cap T| = 2$?

   (d) Let D denote the set of all four-digit number that can be built using the elements of S as digits and allowing repetition of digits. What is $|D|$?

   (e) How many elements of D have four different digits?

   (f) How many elements of D have exactly three different digits?

   (g) How many even numbers belong to D?

   (h) How many members of D have sum of digits 9? For example 1116 and 2331 both qualify. Hint: try the hot-dog Yahtzee model.

7. Four card 'poker' hands. A four-card poker hand is a set of four playing cards selected from a deck of 52 ordinary playing cards (there are four *suits* each with 13 *denominations*.

   (a) How many four-card poker hands are there altogether?

   (b) What is the probability that a randomly selected a four-card poker hand consist entirely of hearts?

   (c) What is the probability that a randomly selected a four-card poker hand consists of two hearts and two clubs?

   (d) What is the probability that a randomly selected a four-card poker hand has three cards of one denomination (value) and one of some other denomination?

8. Recall the definitions of the properties reflexive (R), symmetric (S), transitive (T), and antisymmetric (A). Let $\mathcal{U} = \{1, 2, 3, 4, 5, 6, 7, 8\}$.

   (a) For each relation $\mathcal{R}$ defined on $\mathcal{U}$. List the properties R, S, T, and A satisfied by $\mathcal{R}$. Show your work.

      i. $x\mathcal{R}y$ if $x \leq y$ (in the usual sense).

      ii. $x\mathcal{R}y$ if $(x - y)^2 \leq 1$.

      iii. $x\mathcal{R}y$ if xy is an even number.

      iv. $x\mathcal{R}y$ if $x|y$, ie, if x is a divisor of y.

   (b) Construct the matrix model for the relation in ii.

   (c) Construct the directed graph model for the relation in iv.