December 13, 2000

Your name ______________________________

It is important that you **show your work**.

1. Use the Euclidean algorithm to solve the decanting problem for decanters of sizes 199 and 98. In other words, find integers x and y such that $199x+98y = 1$.

   Solution: Use repeated division and then 'unwind' to get

$$\begin{aligned} 1 &= 3-2 \\ &= 3-(98-3\cdot 32) \\ &= 3+32\cdot 3-98 \\ &= 33\cdot 3-98 \\ &= 33(199-2\cdot 98)-98 \\ &= 33\cdot 199 = 66\cdot 98-98 \\ &= 33\cdot 199-67\cdot 98 \end{aligned}$$

   so $x = 33$ and $y = -67$.

2. Recall that 746 is a decimal representation; that is, a sum of multiples of powers of 10. the word 'interpret' means 'express in decimal notation'.

   (a) Interpret 10110.11_2
   Solution: $2^4 + 2^2 + 2^1 + 2^{-1} + 2^{-2} = 16 + 4 + 2 + 1/2 + 1/4 = 22.75$.

   (b) Interpret 110.11_{-2}
   Solution: $(-2)^2 + (-2)^1 + (-2)^{-1} + (-2)^{-2} = 4 - 2 - 1/2 + 1/4 = 1.75$.

   (c) Find the base 6 representation of 2001.
   Solution: 13133_6

   (d) Find the base 4 representation of 1/9.
   Solution: $0.\overline{013}_4$

3. (Modular Arithmetic) Note that each of the even powers of 10 are congruent to 1 modulo 11; ie, $10^{2n} \equiv 1 (\text{mod } 11)$, while the odd powers of 10 are congruent to -1 modulo 11. You can use these facts to prove, for example, that $1234 \equiv -1+2-3+4 (\text{mod } 11)$. Use the facts above together with the fact that each base-10 number is equivalent to the sum of its digits modulo 9 to

   (a) Find the remainder when 12345678910111213 is divided by 11.

   Solution: The alternating sum of the digits is $1+3+5+7+9+0+1+2+3-(2+4+6+8+1+1+1+1) = 31-24 = 7$, so the remainder is 7.

   (b) Find the remainder when 234567891011121314 is divided by 9.

   Solution: The sum of the digits is $2+3+4+\cdots+4 = 59$ which is congruent to 5 modulo 9, so the remainder is 5.

   (c) Find the remainder when 3456789101112131415 is divided by 99.

   Solution: Find both the sum and the alternating sum of the digits and get $x \equiv 4 (\text{mod } 11)$ and $x \equiv 0 (\text{mod } 9)$. Only 81 has this property.

   (d) Find an ordered pair of digits (a, b) such that $12a6b1$ is divided by 99.

   Solution: Again find the sum and alternating sum of the digits to discover that a and b satisfy $a + b = 8$, so there are nine solutions, $(0, 8), (1, 7), \ldots, (8, 0)$.

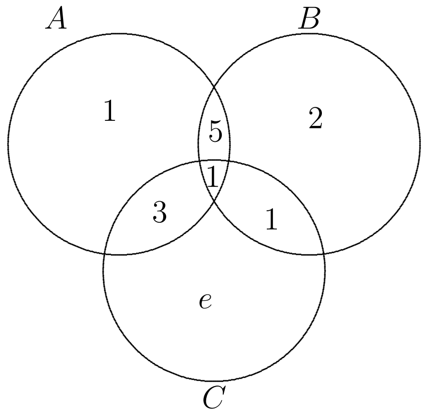
4. Let sets A, B, and C satisfy $|A| = 10, |B| = 9, |C| = c, |AB| = 6,$ $|AC| = 4, |BC| = 2, |ABC| = 1$, and $|A \cup B \cup C| = 23$.

   (a) What is c?

   (b) What is $|\overline{A}\,\overline{B}\,C|$?

   (c) What is $|\overline{AB}C \cup A\overline{BC} \cup \overline{A}B\overline{C}|$?

   Solution: Construct a venn diagram



Then notice that $|A \cup B \cup C| = |A| + |\overline{A}B| + |\overline{AB}C| = 10 + 3 + e = 23$, so $e = 10$ and $c = 10 + 3 + 1 + 1 = 15$. Then $|AB\overline{C} \cup A\overline{B}C \cup \overline{A}BC| = 5 + 3 + 1 = 9$, and $|\overline{AB}C \cup A\overline{BC} \cup \overline{A}B\overline{C}| = 13$.

5. Prove by mathematical induction the formula for sum of the squares of the first n positive integers:

$$1^2 + 2^2 + 3^2 + \cdot + n^2 = \frac{1}{6}n(n+1)(2n+1)$$

Solution: The proposition is $P(n) : 1^2+2^2+3^2+\cdots+n^2 = \frac{1}{6}n(n+1)(2n+1)$. Thus $P(1)$ is $1^2 = \frac{1}{6}(1)(1+1)(2 \cdot 1+1) = 1$. Next assume that $P(n)$ is true for some positive integer n. Then, to prove $P(n+1)$, consider the left side of $P(n+1)$.

$$\begin{aligned} 1^2 + 2^2 + 3^2 + \cdot + n^2 + (n+1)^2 &= \frac{1}{6}n(n+1)(2n+1) + (n+1)^2 \\ &= (n+1)\left(\frac{n(2n+1)}{6} + n + 1\right) \\ &= \frac{1}{6}(n+1)(2n^2+7n+6) \\ &= \frac{1}{6}[(n+1)(n+2)(2n+3)] \\ &= \frac{1}{6}(n+1)(n+2)(2(n+1)+1) \end{aligned}$$

so the inductive step is satisfied. Therefore, but mathematical induction, the $P(n)$ is true for all positive integers n.

6. Let $\mathcal{U} = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$ be the universal set. Let $S = \{1, 2, 3, 4, 5, 6\}$ and $T = \{0, 7, 8, 9\}$. Find each of the following numbers.

   (a) How many subsets does $\mathcal{U}$ have?

   Solution: Since $\mathcal{U}$ has 10 elements, it has 2^{10} subsets.

   (b) How many 5-element subsets does $\mathcal{U}$ have?

   Solution: This is just the number of ways to pick 5 objects from a 10-element set, $\binom{10}{5} = C_5^{10} = 252$.

   (c) How many 5-element subsets A of $\mathcal{U}$ satisfy $|A \cap S| = 3$ and $|A \cap T| = 2$?

   Solution: Pick 3 elements from S and then 2 from T. This can be done in $\binom{6}{3} \cdot \binom{4}{2} = 120$.

   (d) Let D denote the set of all four-digit number that can be built using the elements of S as digits and allowing repetition of digits. What is $|D|$?

   Solution: This is sampling with repetition and order matters. Therefore there are $E_4^6 = 6^4 = 1296$.

   (e) How many elements of D have four different digits?

   Solution: This is sampling without repetition and order matters. Therefore there are $P_4^6 = 6!/(6-4)! = 360$.

   (f) How many elements of D have exactly three different digits?

   Solution: There must be two of one digit and one of two others. So, pick the duplicated digit in one of 6 ways, then pick the other two digits in $\binom{5}{2} = 10$ ways. Then select two locations for the duplicated digit $\binom{4}{2} = 6$ ways, and finally select one of the two orders for the other two digits: $6 \cdot 10 \cdot 6 \cdot 2 = 720$.

   (g) How many even numbers belong to D?

   Solution: Exactly half the 1296 members of D are even numbers, 648.

(h) How many members of D have sum of digits 9? For example 1116 and 2331 both qualify. Hint: try the hot-dog Yahtzee model.

Solution: Imagine 9 vertical bars to be separated into four groups so that each group has $1, 2, 3, 4, 5$, or 6 bars. Note that three dividers works just right, as long as we insist that dividers cannot be adjacent. There are 8 gaps among the bars so the question is 'how many ways can the three dividers be put into the 8 gaps', which is $\binom{8}{3} = 56$.

7. Four card 'poker' hands. A four-card poker hand is a set of four playing cards selected from a deck of 52 ordinary playing cards (there are four *suits* each with 13 *denominations*.

   (a) How many four-card poker hands are there altogether?

   Solution: This is just the number of 4-element subsets of a 52-element set, which is $\binom{52}{4} = 270725$. This is the denominator for all the fractions below.

   (b) What is the probability that a randomly selected a four-card poker hand consist entirely of hearts?

   Solution: The numerator is $\binom{13}{4} = 715$.

   (c) What is the probability that a randomly selected a four-card poker hand consists of two hearts and two clubs?

   Solution: The numerator is $\binom{13}{2} \cdot \binom{13}{2} = 6084$.

   (d) What is the probability that a randomly selected a four-card poker hand has three cards of one denomination (value) and one of some other denomination?

   Solution: To find the numerator, first pick the value of the three of a kind in $\binom{13}{1} = 13$ ways. Then pick three of them in $\binom{4}{1} = 4$ ways. Finally, pick one card from the other 48. Thus we have $13 \cdot 4 \cdot 48 = 2496$.

8. Recall the definitions of the properties reflexive (R), symmetric (S), transitive (T), and antisymmetric (A). Let $\mathcal{U} = \{1, 2, 3, 4, 5, 6, 7, 8\}$.

   (a) For each relation $\mathcal{R}$ defined on $\mathcal{U}$. List the properties R, S, T, and A satisfied by $\mathcal{R}$. Show your work.

   i. $x\mathcal{R}y$ if $x \leq y$ (in the usual sense).

   Solution: R is reflexive because for each x in $\mathcal{U}$, $x \leq x$; not symmetric because $1\mathcal{R}2$ but not $2\mathcal{R}1$; antisymmetric because $x \leq y$ and $y \leq x$ does imply that $x = y$; also $\mathcal{R}$ is transitive since $\leq$ has this property on the set of all real numbers.

   ii. $x\mathcal{R}y$ if $(x - y)^2 \leq 1$.

   Solution: This $\mathcal{R}$ is reflexive and symmetric, but not antisymmetric or transitive. $1\mathcal{R}2 \wedge 2\mathcal{R}1 \wedge 1 \neq 2$ shows that $\mathcal{R}$ is not antisymmetric, and $1\mathcal{R}2 \wedge 2\mathcal{R}3 \wedge \sim (1\mathcal{R}3)$.

   iii. $x\mathcal{R}y$ if xy is an even number.

   Solution: This $\mathcal{R}$ is not reflexive ($\sim (1\mathcal{R}1)$), not antisymmetric ($1\mathcal{R}2 \wedge 2\mathcal{R}1 \wedge 1 \neq 2$), and not transitive ($1\mathcal{R}2 \wedge 2\mathcal{R}1 \wedge \sim (1\mathcal{R}1)$), but it is symmetric, because xy is even if one of them is even.

   iv. $x\mathcal{R}y$ if $x|y$, ie, if x is a divisor of y.

   Solution: Every integer divides itself, so $\mathcal{R}$ is reflexive. $\mathcal{R}$ is not symmetric. It is antisymmetric, however and it is also transitive.

   (b) Construct the matrix model for the relation in ii.

Solution: Order the numbers by value to get the boolean matrix

$$\begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \end{pmatrix}$$

(c) Construct the directed graph model for the relation in iv.

Solution: Given here is the Hasse diagram for the relation. Remember that all the edges of Hasse diagrams are directed upwards. To get the digraph, add loops at each vertex and add the edges that result from transitivity.