May 9, 2001

Your name ______________________________

It is important that you **show your work**.

1. Let $\mathcal{U} = \{1, 2, \ldots, 20\}$ be the universal set. Let $A = \{1, 2, \ldots, 10\}$, $B = \{11, 12, \ldots, 20\}$ and $C = \{8, 9, 10, 11, 12\}$. Find the following cardinalities

   (a) $|(A \times A) \cup (B \times B) \cup (C \times C)|$
   Solution: Use inclusion-exclusion to get $100 + 100 + 25 - 9 - 4 = 212$.

   (b) $|(\mathcal{U} \times \mathcal{U}) - (A \times A)|$
   Solution: $400 - 100 = 300$.

   (c) $|(A \cap C) \times (B \cap C)|$
   Solution: $3 \cdot 2 = 6$.

   (d) How many subsets S of $\mathcal{U}$ satisfy both $|S \cap A| = 2$ and $|S \cap B| = 3$?
   Solution: $\binom{10}{2} \cdot \binom{10}{3} = 5400$.

   (e) How many subsets S of $\mathcal{U}$ satisfy $|S \cap A| = |S \cap B| = |S \cap C| = 2$? For example, $\{2, 8, 11, 15\}$ and $\{9, 10, 14, 15\}$ both satisfy these conditions.
   Solution: There are three types of sets. A. those S that have 2 members in the set $\{8, 9, 10\}$, B. those with one member, and C. those with none. The number of such subsets is A. $\binom{3}{2} \cdot \binom{8}{2} = 84$, B. $7 \cdot 3 \cdot 2 \cdot 8 = 336$; and C.$\binom{7}{2} \cdot \binom{3}{2} = 21$, for a total of 441.

   (f) $|A \times B \times C|$
   Solution: $|A \times B \times C| = |A| \cdot |B| \cdot |C| = 10 \cdot 10 \cdot 5 = 500$.

2. Let $A = \{1, 2, 3, 4\}$

   (a) How many relations on A are there? It may help to think of a relation on A as a 4 by 4 Boolean matrix.
   Solution: There are 16 'yes', 'no' decisions to be made so there are $2^{16} = 65536$ relations.

   (b) How many relations on A are reflexive?
   Solution: There are 12 entries that can be either 0 or 1, so there are 2^{12} reflexive relations.

   (c) How many relations on A are symmetric?
   Solution: 2^{10}.

   (d) How many relations on A are both reflexive and symmetric?
   Solution: 2^6.

(e) How many relations on A are reflexive, symmetric and transitive?

Solution: This is much harder. Since every relation satisfying RS and T is an equivalence relation, we can could the number of ways to partition the set A. Brute force gets 15 of these.

(f) Let $R = \{(1,2),(2,3),(3,4),(2,1)\}$. How many ordered pairs belong to the smallest transitive relation R_t that contains R. The relation R_t is called the transitive closure of R. The question is, what is $|R_t|$?

Solution: $R_t = \{(1,2),(2,3),(3,4),(2,1),(1,1),(2,2),(1,3),(1,4),(2,4)\}$, so $|R_t| = 9$.

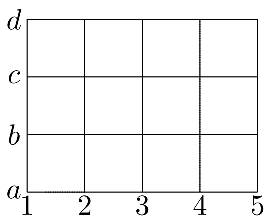
3. (a) Show that among any nine points $P_1, P_2, P_3, \ldots, P_9$ in space, where $P_i = (x_i, y_i, z_i)$ and $0 \le x_i \le 2,\ 0 \le y_i \le 2,\ 0 \le z_i \le 2$, there is some pair whose distance apart is at most 1.75. In other words, if you squeeze 9 points into a $2 \times 2 \times 2$ box, at least two of them must end up fairly close to each other. Recall that the distance between $(0,0,0)$ and $(1,1,1)$ is given by $d = \sqrt{1^2+1^2+1^2} = \sqrt{3} \approx 1.73$.

   Solution: The given box can be partitioned into 8 unit boxes (some with their surfaces, some not), so with nine points to distribute, some unit box must get two points. There two points can be at most $\sqrt{3} < 1.75$ apart.

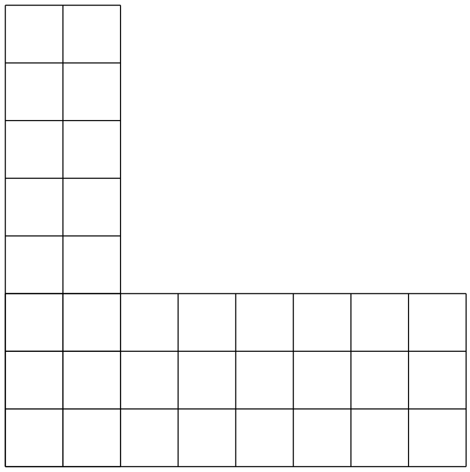
   (b) Show that the result is not true if only eight points are selected.

   Solution: Partition the cube into eight unit cubes, which serve as pigeonholes. Since there are nine points, some unit cube must contain at least two of the points. These two points are at most $\sqrt{3} < 1.75$ units apart. For the second part, the eight points could be $(0,0,0),(0,0,2),(0,2,0),$ $(2,0,0),(2,2,0),(2,0,2),(0,2,2)$, and $(2,2,2)$. No two of these are with 2 of each other.

4. Counting Rectangles. Consider the 3×4 grid of squares shown:

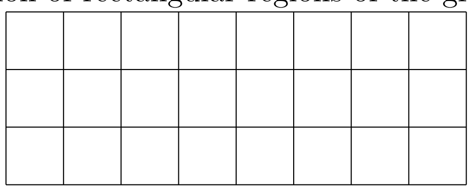


To count the number of rectangle bounded by gridlines, notice that each rectangle is determined by its top and bottom lines together with its left and right bounding lines. For example, $\{a,d\},\{2,4\}$ determines the middle 3 by 2 rectangle in the figure. Use this idea to show that there are 60 rectangular regions. Let P denote the set of rectangular subregions of the grid



The problem is to count the number of rectangular subregions of grid P by first counting the rectangles in certain regions of P.

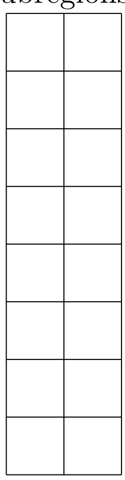
(a) Let A denote the collection of rectangular regions of the grid



How many rectangular regions are bounded by gridlines in A?

Solution: There are $\binom{4}{2} = 6$ ways to pick the upper and lower boundaries, and $\binom{9}{2} = 36$ ways to pick the left and right boundaries, so there are 216 rectangular regions bounded by the gridlines.

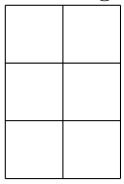
(b) Let B denote the set of rectangular subregions of



Find $|B|$.

Solution: There are $\binom{9}{2} = 36$ ways to pick the upper and lower boundaries, and $\binom{3}{2} = 3$ ways to pick the left and right boundaries, so there are 108 rectangular regions bounded by the gridlines.

(c) Finally, let C denote the collection of rectangular subregions of the grid



Compute $|C|$.

Solution: Notice that $C = A \cap B$. There are $\binom{4}{2} = 6$ ways to pick the upper and lower boundaries, and $\binom{3}{2} = 3$ ways to pick the left and right boundaries, so there are 18 rectangular regions bounded by the gridlines.

(d) Use the inclusion-exclusion principle to find $|P|$ in terms of $|A|, |B|$, and $|C|$.

Solution: According to the inclusion-exclusion principle $|P| = |A| + |B| - |C| = 216 + 108 - 18 = 306$.

5. The number $N = 5^{28} \cdot 7^{12}$ has 30 digits (in decimal notation). Prove that some digit appears at least four times in the representation as follows:

(a) Carefully write out the negation of the conclusion.

Solution: If no digit appears four times, then every digit appears exactly three times.

(b) If the negation is true what would be the sum of the digits.

Solution: In case every digit appears exactly three times, the sum of the 30 digits is $3(0 + 1 + 2 \ldots + 9) = 3 \cdot 45 = 135$.

(c) What would the truth of the negation imply about the divisibility of N by 9.

Solution: If the sum of the digits is 135, then the number must be a multiple of 9.

(d) Why can N not be a multiple of 9?

Solution: The number N cannot be a multiple of 9 because its factorization into primes would include at least two 3's, but we already know the factorization is $N = 5^{28} \cdot 7^{12}$.

6. Use the Euclidean algorithm to solve the decanting problem for decanters of sizes 217 and 475. In other words, find integers x and y such that $gcd(217, 475) = 217x + 475y$.

Solution: Repeatedly divide, starting with 217 divided into 475. Then unwind the process to get

$$\begin{aligned} 1 &= 5 - 2 \cdot 2 = 5 - 2(12 - 2 \cdot 5) \\ &= 5 - 2 \cdot 12 \end{aligned}$$

$$
\begin{aligned}
&= 5(41 - 3 \cdot 12) - 2 \cdot 12 \\
&= 5 \cdot 41 - 17 \cdot 12 \\
&= 5 \cdot 41 - 17(217 - 5 \cdot 41) \\
&= 90 \cdot 41 - 17 \cdot 217 \\
&= 90(475 - 2 \cdot 217) - 17 \cdot 217 \\
&= 90 \cdot 475 - 197 \cdot 217
\end{aligned}
$$

So we have the solution $x = -197, y = 90$.

7. (Coin flips/probability) A fair coin is flipped seven times. The sample space for this experiment is given by

$$S = \{(x_1, x_2, x_3, x_4, x_5, x_6, x_7) \mid x_i \in \{H, T\} \text{ for } i = 1, 2, \ldots 7\}.$$

Of course, S has $|S| = 2^7 = 128$ elements. Let $A_1 = \{(x_1, x_2, x_3, x_4, x_5, x_6, x_7) \mid x_1 = H\}$, $A_2 = \{(x_1, x_2, x_3, x_4, x_5, x_6, x_7) \mid x_2 = H\}$, and in general

$$A_i = \{(x_1, x_2, x_3, x_4, x_5, x_6, x_7) \mid x_i = H\},$$

for $i = 1, 2, \ldots 7$. In other words, A_i is the event that the i^{th} flip results in heads. Let $P(E)$ denote the probability of event E.

(a) What is $|A_1|$, and what is $P(A_1)$?
Solution: $|A_1| = 2^6 = 64$ and $P(A_1) = 1/2$.

(b) What is $|A_1 A_2 A_3|$?
Solution: $|A_1 A_2 A_3| = 2^4 = 16$.

(c) What is the probability that exactly three of the seven flips result in heads. Your answer is the number p_3 referred to in the next part.
Solution: $P(\text{exactly three heads}) = \binom{7}{3} \cdot 1/128 = 35/128$.

(d) Find the probability p_i of getting exactly i heads in the seven flips, for $i = 0, 1, 2, 3, 4, 5, 6$, and 7.
Solution: $p_0 = 1/128, p_1 = 7/128, p_2 = 21/128, p_3 = 35/128, p_4 = 35/128, p_5 = 21/128, p_6 = 7/128,$ and $p_7 = 1/128$.

8. John has 1 penny, 3 nickels, 2 dimes, 3 quarters, and 2 dollars. For how many different amounts can John make an exact purchase (with no change required)? Hint notice that some amounts can be made in several ways.

Solution: We'll count achievable amounts in pairs: first lets include 0 for the time being. $0, 1, 5, 6, 10, 11, 15, 16, \ldots, 310, 311$. Its clear that there are 63 pairs of numbers in the list, for a total of 126. Throwing out 0 gives 125 numbers.

9. (a) Give an example that shows that the union $R \cup T$ of two transitive relations R and T on the set A need not be transitive.

   Solution: One easy example is $R = \{(1,2)\}$ and $T = \{(2,3)\}$.

   (b) Recall that for any relation R on a set A, the relation $R^{-1} = \{(y,x) | \, (x,y) \in R\}$. In other words, to get R^{-1} flip all the pairs of R around. Prove that if a relation R is transitive, then so is R^{-1}.

   Solution: Suppose $xR^{-1}y \wedge yR^{-1}z$. Then yRx and zRy. In other words, $zRy \wedge yRx$, which implies that zRx. This is the same as saying that $xR^{-1}z$ which is what we needed to prove.

   (c) Prove that the compliment $\overline{R}$ of a symmetric relation R is also symmetric.

   Solution: Suppose $x\overline{R}y$. Then (x,y) is not in R, and so (y,x) is not in R (otherwise R would not be symmetric). Thus $y\overline{R}x$, which proves symmetry of $\overline{R}$.

10. Five cards are selected at random from a deck of 52 ordinary playing cards (there are four *suits* each with 13 *denominations*. Let $P(E)$ denote the probability of event E.

   (a) What is the probability that all five cards selected are red?

   Solution: Throughout this solution, let $N = C_5^{52} = \binom{52}{5} = 2,598,960$. Each one of the fractions in this problem has N as denominator. Here the numerator is $\binom{26}{5} = 65780$ so the probability is approximately 0.02531.

   (b) What is the probability that all five cards selected are hearts?

   Solution: $C_5^{13} = 1287$, and division by N yields 0.0004951.

   (c) What is the probability that there are three hearts and two clubs?

   Solution: $C_3^{13} \cdot C_2^{13} = 78 \cdot 286 = 22308$, and division by N yields 0.0085834.

   (d) What is the probability that each of the four suits are represented among the five cards?

   Solution: First, choose the suit that will have two cards, then pick two cards from that suit. Then pick one card from each of the other suits to get $4 \cdot \binom{13}{2} \cdot 13^3 = 685464$ and the probability is 0.26374.

   (e) Let E_i denote the event that exactly i of the five cards are red. Find $P(E_0), P(E_1), P(E_2), P(E_3), P(E_4)$ and $P(E_5)$?

   Solution: In order, these numbers are $P(E_0) = \binom{26}{5} \cdot \binom{26}{0} \div N \approx 0.02513$, $P(E_1) = \binom{26}{4} \cdot \binom{26}{1} \div N \approx 0.14955$, $\binom{26}{3} \cdot \binom{26}{2} \div N \approx 0.32513$, $\binom{26}{2} \cdot \binom{26}{3} \div N \approx 0.32513$, $\binom{26}{1} \cdot \binom{26}{4} \div N \approx 0.14955$, and $\binom{26}{0} \cdot \binom{26}{5} \div N \approx 0.02513$.

11. Find the base 4 representation of each of the following numbers.

    (a) The decimal (ie, base ten) numeral 2002

    Solution: Repeated division yields 133102_4

    (b) $2^9 + 2^7 + 2^5 + 2^3 + 1$

    Solution: Convert to binary first to get 1010101001_2 which is easily converted to base 4: 22221_4.

    (c) $3 \cdot 16^3 + 5 \cdot 16 + 11 \cdot 16^{-2}$

    Solution: Of course the base 16 representation is $3050.0b_{16}$. Each base 16 digit gives rise to two base four digits. For example the digit 5 converts to 11_4 and the digit $b = 11$ gives rise to 23_4. Thus we get 3001100.0023_4.

    (d) The decimal fraction 0.7.

    Solution: Repeated multiplication yields the repeating expression $0.2\overline{30}$.

    (e) Explain how you can find the base 2 representation of a base 4 numeral without converting it into a decimal first.

    Solution: Each base 4 numeral give rise to two binary digits and conversely. For example, $110011110_2 = 1\ 11\ 01\ 11\ 10_2 = 13132_4$.