

June 27, 2001

Your name _____

It is important that you **show your work**. The total value of this test is 220 points.

1. (10 points) Use the Euclidean algorithm to solve the decanting problem for decanters of sizes 1317 and 1075. In other words, find integers x and y such that $\gcd(1317, 1075) = 1317x + 1075y$.

2. (15 points) Prove by mathematical induction the formula for sum of the cubes of the first n positive integers:

$$1^3 + 2^3 + 3^3 + \cdots + n^3 = (n(n+1)/2)^2.$$

In other words, the sum of the cubes of the first n positive integers is the square of the sum of the first n positive integers. Write down explicitly the first five equations.

3. (18 points) Let $S = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$ be the set of nonzero digits. Let D denote the set of all three-digit numbers that can be built using the elements of S as digits and allowing repetition of digits.

(a) What is $|D|$?

(b) How many elements of D have three different digits?

(c) How many elements of D are multiples of 99?

(d) How many elements of D are multiples of 3?

(e) How many elements of D have exactly two different digits?

(f) How many even numbers belong to D ?

4. (20 points) Let $\mathcal{U} = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$ be the universal set. Let $S = \{1, 2, 3, 4, 5, 6\}$ and $T = \{6, 7, 8, 9\}$. Find each of the following numbers.

(a) How many subsets does \mathcal{U} have?

(b) How many 5-element subsets does \mathcal{U} have?

(c) How many subsets A of \mathcal{U} satisfy $|A \cap S| = 4$ and $|A \cap T| = 2$? Give an example of such a set with 6 as a member and one that does not have 6 as a member.

(d) How many subsets of \mathcal{U} have an even number of elements?

(e) What is the cardinality of $\mathcal{U} \times \mathcal{U} - (S \times T)$?

6. (20 points)

(a) Prove that the intersection of two transitive relations on the set A is also transitive.

(b) Prove that the union of two symmetric relations on the set A is also symmetric.

(c) Prove that the complement \overline{R} of a symmetric relation R on the set A is symmetric.

(d) Give an example that shows that the union of two antisymmetric relations on the set A need not be antisymmetric.

7. (20 points) Let Z denote the set of all integers. Define R on Z by xRy if $x - y$ is a multiple of 5 (note that 0 is a multiple of 5). Which of the following properties does R satisfy? Give *reasons* for each answer. The reason is roughly four times the value of the correct yes-no answer.

(a) reflexivity

(b) symmetry

(c) transitivity

(d) antisymmetry

(e) Find the cells of R . Is R an equivalence relation?

8. (20 points) Bridge hands. A 13-card bridge hand is a set of 13 playing cards selected from a deck of 52 ordinary playing cards (there are four *suits* each with 13 *denominations*).

(a) How many 13-card bridge hands are there altogether?

(b) How many 13-card bridge hands consist of five hearts, four clubs, and four spades?

(c) How many 13-card bridge hands consist entirely of hearts and spades?

(d) How many 13-card bridge hands have distribution $5 - 4 - 3 - 1$?

(e) How many 13-card bridge hands have exactly two suits represented?

9. (15 points) Find the base 9 representation of each of the following numbers.

(a) 2001

(b) $3^9 + 3^7 + 3^5 + 3^3 + 1$

(c) $4 \cdot 27^3 + 2 \cdot 27^2 + 19 \cdot 27^{-1}$

(d) Explain how you can find the base 9 representation of a base 3 numeral without converting it into a decimal first.

10. (20 points) Recall that a Yahtzee Roll is a roll of five indistinguishable dice.

(a) How many different Yahtzee Rolls are possible?

(b) Each Yahtzee Roll has a *pattern*, ie, a string of letters that describes the number of duplicates that appear. For example, we might say the rolls $\{2, 2, 3, 3, 4\}$ and $\{1, 3, 4, 3, 1\}$ both have the pattern *aabbc*. How many different patterns are there?

(c) For each pattern in (b), find the number of Yahtzee rolls.

11. (10 points) What is the smallest positive integer multiple of 99 that has exactly 16 positive integer divisors? Recall that the number of divisors of $2^i 3^j 5^k 7^m$, for example, is $(i + 1)(j + 1)(k + 1)(m + 1)$.
12. (12 points) Let $I = [0, 1]$, the unit interval of real numbers. Let $J = [0, 1] \times [0, 1] \times [0, 1]$, the unit cube in 3-space. Define a mapping of I onto J that is one-to-one. Show that your mapping is onto.

13. (10 points) Show that the set $A = \{2, 4, 6, 8, \dots\}$ of positive even integers is equivalent (in the sense of Cantor) to the set Z of all integers. The important part of this problem is to define the bijection between the two sets and to show that it is both 1-1 and onto.