Math 1165 **Discrete Math** **Final Exam**

June 27, 2001

Your name ______________________________

It is important that you **show your work**. The total value of this test is 220 points.

1. (10 points) Use the Euclidean algorithm to solve the decanting problem for decanters of sizes 1317 and 1075. In other words, find integers x and y such that $gcd(1317, 1075) = 1317x + 1075y$.
   Solution: Use repeated division, etc. to find that $x = -422$ and $y = 517$.

2. (15 points) Prove by mathematical induction the formula for sum of the cubes of the first n positive integers:

$$1^3 + 2^3 + 3^3 + \cdots + n^3 = (n(n+1)/2)^2.$$

   In other words, the sum of the cubes of the first n positive integers is the square of the sum of the first n positive integers. Write down explicitly the first five equations.
   Solution: The proposition is $P(n) : 1^3 + 2^3 + 3^3 + \cdots + n^3 = (n(n+1)/2)^2$. Thus $P(1)$ is $1^3 = ((1)(1+1)/2)^2 = 1$. Next assume that $P(n)$ is true for some positive integer n. Then, to prove $P(n+1)$, consider the left side of $P(n+1)$.

$$\begin{aligned} 1^3 + 2^3 + 3^3 + \cdot + n^3 + (n+1)^3 &= (n(n+1)/2)^2 + (n+1)^3 \\ &= (n+1)^2\left(\frac{n^2}{4} + n + 1\right) \\ &= \frac{1}{4}(n+1)^2(n^2+4n+4) \\ &= \frac{1}{4}[(n+1)^2(n+2)^2] \\ &= [(n+1)(n+2)/2]^2 \end{aligned}$$

   so the inductive step is satisfied. Therefore, but mathematical induction, the $P(n)$ is true for all positive integers n.

3. (18 points) Let $S = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$ be the set of nonzero digits. Let D denote the set of all three-digit numbers that can be built using the elements of S as digits and allowing repetition of digits.

   (a) What is $|D|$?
   Solution: This is sampling with repetition and order matters. Therefore there are $E^9_3 = 9^3 = 729$.

   (b) How many elements of D have three different digits?

   Solution: This is sampling without repetition and order matters. Therefore there are $P^9_3 = 6!/(9-3)! = 504$.

(c) How many elements of D are multiples of 99?
Solution: List them. All the positive multiples of 99 less than 1000 use nonzero digits except 990, so the answer is 8.

(d) How many elements of D are multiples of 3?
Solution: Oddly, this number is just one third the size of D, so there are 243 such numbers. The best way to see this is to note that there are 3 digits in each of the categories $x \equiv 0(\mod 3)$, $x \equiv 1(\mod 3)$, $x \equiv 2(\mod 3)$, and arguing a number with three digits from just one category or a number whose three digits are all in different categories will work. There are $3 \cdot 3^3$ of the first type and $6 \cdot 3^3$ of the second.

(e) How many elements of D have exactly two different digits?
Solution: There must be two of one digit and one of the other. So, pick the duplicated digit in one of 9 ways, then pick the other digit in one of eight ways. Then select two locations for the duplicated digit $\binom{3}{2} = 3$ ways. The final count is $9 \cdot 2 \cdot 8 = 144$.

(f) How many even numbers belong to D?
Solution: Exactly four ninth's of D are even numbers, 324.

4. (20 points) Let $\mathcal{U} = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$ be the universal set. Let $S = \{1, 2, 3, 4, 5, 6\}$ and $T = \{6, 7, 8, 9\}$. Find each of the following numbers.

(a) How many subsets does $\mathcal{U}$ have?

Solution: Since $\mathcal{U}$ has 9 elements, it has 2^9 subsets.

(b) How many 5-element subsets does $\mathcal{U}$ have?
Solution: This is just the number of ways to pick 5 objects from a 9-element set, $\binom{9}{5} = C^9_5 = 126$.

(c) How many subsets A of $\mathcal{U}$ satisfy $|A \cap S| = 4$ and $|A \cap T| = 2$? Give an example of such a set with 6 as a member and one that does not have 6 as a member.
Solution: There are two types, those with 6 and those without. If $6 \in A$, we can pick 3 more members of S and one more in T in $10 \cdot 3 = 30$ ways. If 6 in not in A, we must choose 4 elements from S and then 2 from T. This can be done in $\binom{5}{4} \cdot \binom{3}{2} = 15$. The total is therefore 45 such sets.

(d) How many subsets of $\mathcal{U}$ have an even number of elements?
Solution: Half the subsets of $\mathcal{U}$ have an even number of elements.

(e) What is the cardinality of $\mathcal{U} \times \mathcal{U} - (S \times T)$?
Solution: $81 - 24 = 57$

5. (30 points) Let $A = \{1, 2, 3, 4\}$.

(a) How many relations on A are there?
Solution: $2^{16} = 65536$

(b) Find a relation R on A that has exactly 3 ordered pair members and is both symmetric and antisymmetric.
Solution: The matrix $\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$ is the matrix of a symmetric and antisymmetric relation.

(c) Prove that every relation R on A with 15 ordered pair members is not transitive.
Solution: Suppose such a relation is transitive and (x, y) is the ordered pair that does not belong to R. Take z different from x and y. Then xRz and zRy together with transitivity imply that xRy, a contradiction.

(d) Find an equivalence relation R on A that has exactly 10 elements.
Solution: The matrix below is that of a ten member equivalence relation.
$$\begin{pmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

(e) Find a transitive, non-reflexive, non-symmetric, non-antisymmetric relation R on A that has exactly 6 elements.
Solution: The matrix below is that of such a relation.
$$\begin{pmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

(f) How many relations R on A have exactly seven ordered pair members? How many of these have exactly one loop? How many of these have exactly two loops?
Solution: There are 16 positions in the matrix to be determined. There are $\binom{16}{7} = 11440$ ways to choose these seven positions. The number with exactly one loop is $\binom{4}{1} \cdot \binom{12}{6} = 4 \cdot 924 = 3696$.

6. (20 points)

(a) Prove that the intersection of two transitive relations on the set A is also transitive.
Solution: Let R and S be transitive relations on the set A. To prove that $R \cap S$ is also transitive, suppose $xR \cap Sy$ and $yR \cap Sz$. Then xRz and xSz because each of the relations is transitive. Hence $xR \cap Sz$. Thus $R \cap S$ is transitive.

(b) Prove that the union of two symmetric relations on the set A is also symmetric.
Solution: Let R and S be symmetric relations on the set A. To prove that $R \cup S$ is also symmetric, suppose $xR \cup Sy$. Then xRy or xSy by the meaning of union. Since R and S are both symmetric, it follows that either yRx or ySx. But this is just what is needed to prove that $yR \cup Sx$. Thus $R \cup S$ is symmetric.

(c) Prove that the compliment $\overline{R}$ of a symmetric relation R on the set A is symmetric.
Solution: Suppose R is symmetric. To prove $\overline{R}$ is symmetric, we use the definition of symmetry. Suppose $x\overline{R}y$. Then (x, y) does not belong to R. If (y, x) belongs to R, then by symmetry of R, the ordered pair (x, y) would have to belong as well. Therefore (y, x) does not belong to R. But this means that (y, x) does belong to $\overline{R}$. Thus $\overline{R}$ is symmetric, by the definition of symmetry.

(d) Give an example that shows that the union of two antisymmetric relations on the set A need not be antisymmetric.
Solution: The two matrices below come from antisymmetric relations, but the union of the two is not antisymmetric.
$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$

7. (20 points) Let Z denote the set of all integers. Define R on Z by xRy if $x - y$ is a multiple of 5 (note that 0 is a multiple of 5). Which of the following properties does R satisfy? Give *reasons* for each answer. The reason is roughly four times the value of the correct yes-no answer.

(a) reflexivity
Solution: This follows from the fact that 0 is a multiple of 5.

(b) symmetry
Solution: If $x - y$ is a multiple of 5, then $y - x$ is also a multiple of 5.

(c) transitivity
Solution: If $x - y$ and $y - z$ are both multiples of 5 then so is their sum $x - y + (y - z) = x - z$.

(d) antisymmetry
Solution: The relation is not antisymmetric because $2R7 \wedge 7R2$ but $2 \neq 7$.

(e) Find the cells of R. Is R an equivalence relation?
Solution: Yes, R is an equivalence relation and the cells of R partition the integers.
$[0] = \{0, \pm 5, \pm 10, \pm 15, \ldots\}$
$[1] = \{1, 1 + \pm 5, 1 + \pm 10, 1 + \pm 15, \ldots\}$

$[2] = \{2, 2 + \pm 5, 2 + \pm 10, 2 + \pm 15, \dots\}$
$[3] = \{3, 3 + \pm 5, 3 + \pm 10, 3 + \pm 15, \dots\}$
$[4] = \{4, 4 + \pm 5, 4 + \pm 10, 4 + \pm 15, \dots\}$

8. (20 points) Bridge hands. A 13-card bridge hand is a set of 13 playing cards selected from a deck of 52 ordinary playing cards (there are four *suits* each with 13 *denominations*).

   (a) How many 13-card bridge hands are there altogether?
   Solution: $\binom{52}{13} = 635013559600$.

   (b) How many 13-card bridge hands consist of five hearts, four clubs, and four spades?
   Solution: $\binom{13}{5} \cdot \binom{13}{4} \cdot \binom{13}{4} = 1287 \cdot 715 \cdot 715 = 657946575$.

   (c) How many 13-card bridge hands consist entirely of hearts and spades?
   Solution: $\binom{26}{13} = 10400600$.

   (d) How many 13-card bridge hands have distribution $5 - 4 - 3 - 1$?
   Solution: First you must select the suits to have 5, 4, 3, and 1 cards in, then pick the right number of cards from that suit: $P_4^4 \cdot \binom{13}{5} \cdot \binom{13}{4} \cdot \binom{13}{3} \cdot \binom{13}{1} = 24 \cdot 1287 \cdot 715 \cdot 286 \cdot 13 = 82111732560$.

   (e) How many 13-card bridge hands have exactly two suits represented?
   Solution: There are $\binom{26}{13} = 10400600$ hands consisting only of hearts and spades, and there are $\binom{4}{2} = 6$ ways to pick a pair of suits, but the all heart hand gets triple counted, so we must subtract $4 \cdot 2 = 8$ from our total to get $6 \cdot 10400600 - 8 = 62403592$.

9. (15 points) Find the base 9 representation of each of the following numbers.

   (a) 2001
   Solution: Repeated division gets $2001 = 2663_9$.

   (b) $3^9 + 3^7 + 3^5 + 3^3 + 1$
   Solution: Repeatedly rewrite the expression as follows:

$$\begin{aligned} 3^9 + 3^7 + 3^5 + 3^3 + 1 &= 3 \cdot 3^8 + 3 \cdot 3^6 + 3 \cdot 3^4 + 3 \cdot 3^2 + 1 \\ &= 3 \cdot 9^4 + 3 \cdot 9^3 + 3 \cdot 9^2 + 3 \cdot 9 + 1 \\ &= 33331_9 \end{aligned}$$

   (c) $4 \cdot 27^3 + 2 \cdot 27^2 + 19 \cdot 27^{-1}$
   Solution: Repeatedly rewrite the expression as follows:

$$\begin{aligned} 4 \cdot 27^3 + 2 \cdot 27^2 + 19 \cdot 27^{-1} &= 4 \cdot 3^9 + 2 \cdot 3^6 + 19 \cdot 3^{-3} \\ &= 12 \cdot 9^4 + 2 \cdot 9^3 + 19 \cdot 3 \cdot 9^{-2} \end{aligned}$$

$$\begin{aligned} &= 9\cdot 9^4 + 3\cdot 9^4 + 2\cdot 9^3 + 57\cdot 9^{-2} \\ &= 1\cdot 9^5 + 3\cdot 9^4 + 2\cdot 9^3 + 6\cdot 9\cdot 9^{-2} + 3\cdot 9^{-2} \\ &= 132000.63_9 \end{aligned}$$

(d) Explain how you can find the base 9 representation of a base 3 numeral without converting it into a decimal first.
Solution: You can group the digits in pairs starting at the radix point. Convert each base 3 pair into its equivalent base 9 digit. For example the number 122021_3 is transformed into 567_9 because $12_3 = 5_9; 20_3 = 6_9$; and $21_3 = 7_9$.

10. (20 points) Recall that a Yahtzee Roll is a roll of five indistinguishable dice.

(a) How many different Yahtzee Rolls are possible?
Solution: The answer is $Y_5^6 = \binom{6+5-1}{5} = 252$.

(b) Each Yahtzee Roll has a *pattern*, ie, a string of letters that describes the number of duplicates that appear. For example, we might say the rolls $\{2,2,3,3,4\}$ and $\{1,3,4,3,1\}$ both have the pattern *aabbc*. How many different patterns are there?
Solution: There are seven patterns. See below for the list.

(c) For each pattern in (b), find the number of Yahtzee rolls.
Solution: The patterns seven are *aaaaa*, *aaaab*, *aaabb*, *aaabc*, *aabbc*, *aabcd*, and *abcde*. The number of rolls for each of these is

i. *aaaaa*: 6
ii. *aaaab*: 30
iii. *aaabb*: 30
iv. *aaabc*: 60
v. *aabbc*: 60
vi. *aabcd*: 60
vii. *abcde*: 6, for a total of $6 + 30 + 30 + 60 + 60 + 60 + 6 = 252$.

11. (10 points) What is the smallest positive integer multiple of 99 that has exactly 16 positive integer divisors? Recall that the number of divisors of $2^i 3^j 5^k 7^m$, for example, is $(i+1)(j+1)(k+1)(m+1)$.
Solution: It should be a number of the form $p^i \cdot 3^j \cdot 11$, where $j \geq 2$ and $(i+1)(j+1)2 = 16$, so try $2\cdot 3^3 \cdot 11 = 594$.

12. (12 points) Let $I = [0,1]$, the unit interval of real numbers. Let $J = [0,1]\times[0,1]\times[0,1]$, the unit cube in 3-space. Define a mapping of I onto J that is one-to-one. Show that your mapping is onto.
Solution: If $x \in [0,1]$ has representation $x = .x_1x_2x_3\ldots$, where the x_i are digits, define $f(x)$ to be the triplet $(.x_1x_4x_7\ldots, .x_2x_5x_8\ldots, .x_3x_6x_9\ldots)$. There is

a problem with dual representation of numbers whose decimal representations end in zeros, but we will not attempt to overcome this problem. Sidestepping this problem, we can 'prove' that the function f defined above is both one-to-one and onto.

13. (10 points) Show that the set $A = \{2, 4, 6, 8, \ldots\}$ of positive even integers is equivalent (in the sense of Cantor) to the set Z of all integers. The important part of this problem is to define the bijection between the two sets and to show that it is both 1-1 and onto.
Solution: Define a function $f : Z \to A$ by

$$f(x) = \begin{cases} 4n + 2 & \text{if } n \geq 0 \\ -4n & \text{if } n < 0 \end{cases}$$

Thus $f(0) = 2, f(-1) = 4, f(1) = 6, f(-2) = 8$ and $f(2) = 10$. Clearly $f(n) \in A$ for each $n \in Z$. To see that f is one-to-one, suppose $m < n$ are integers. We need to show that $f(m) \neq f(n)$. If both m and n are nonnegative, then $f(m) = 4m+2 < 4n+2 = f(n)$. If both are negative, then $fm) = -4m > -4n = f(n)$. If m is negative and n is positive, then $f(m)$ is divisible by 4 and $f(n)$ is not. To see that f is onto, let $b \in A$. Then either $b \equiv 0(\mod 4)$ or $b \equiv 2(\mod 4)$. If $b \equiv 0(\mod 4)$, then $f(-b/4) = -4(-b/4) = b$ and if $b \equiv 2(\mod 4)$, then $f(\frac{b-2}{4}) = 4(\frac{b-2}{4}) + 2 = b$, so f is onto. This completes the proof.