Math 1165 **Discrete Math** **Final Exam**

May 9, 2003 **Show all work** **Name** ______________________________

There are 260 points available on this test.

1. (15 points) Solve the decanting problem for containers of sizes 199 and 179; that is find integers x and y satisfying

$$199x + 179y = d$$

where d is the GCD of 199 and 179.

2. (15 points) Find digits a, b, and c (between 0 and 4) such that $abc_5 = cba_8$, or prove that there are none.

3. (15 points) It is known that the exponential function $b^x, b > 1$ eventually grows larger than any polynomial. In particular, $2^n > n^3$ for all n larger than some threshold value N.

   (a) Find an integer N such that $2^n > n^3$ for all $n \geq N$.

   (b) Use the Principle of Mathematical Induction to prove that your value of N is correct; that is, $2^n > n^3$ for all $n \geq N$. Note that $(n+1)^3 = n^3 + 3n^2 + 3n + 1$.

4. (15 points) The sequence of Fibonacci numbers $f_0, f_1, f_2, \ldots$ is defined by the rule $f_0 = 0$, $f_1 = 1$, and $f_n = f_{n-1} + f_{n-2}$ for $n \geq 2$. Prove that $f_0 + f_2 + f_4 + f_6 + \cdots + f_{2n} = f_{2n+1} - 1$ for all $n \geq 0$.

5. (20 points) Suppose $S = \{1, 2, \ldots, 20\}$. A subset T of size 3 is randomly selected.

   (a) Find the probability that T consists of two odd numbers and one even number.

   (b) Find the probability that T consists of three prime numbers.

   (c) Find the probability that the three numbers in T have a sum that is less than 9.

   (d) Find the probability that T has at least one even number in it.

6. (35 points) A discrete math class has 10 women and 7 men.

   (a) How many 5 element subsets does the class have?

   (b) How many ways are there to choose a committee of size 5 consisting entirely of women?

   (c) How many ways are there to choose a committee of size 5 consisting of 4 women and 1 man?

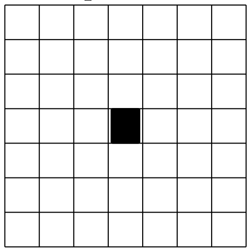
   (d) How many ways are there to choose a committee of size 5 consisting of 3 women and 2 men?

   (e) How many ways are there to choose a committee of size 5 consisting of 2 women and 3 men?

   (f) How many ways are there to choose a committee of size 5 consisting of 1 woman and 4 men?

   (g) How many ways are there to choose a committee of size 5 consisting of 5 men?

7. (20 points) Consider the grid of unit squares below.



   (a) How many square regions are bounded the grid lines?

   (b) How many rectangular regions are bounded by grid lines?

   (c) How many square regions containing the shaded square are bounded the grid lines?

   (d) How many rectangular regions containing the shaded square are bounded by grid lines?

8. (35 points) Let $S = \{1, 2, 3, 4, 5, 6, 7\}$.

   (a) How many non-empty subsets does S have?

   (b) How many subsets of S have no even numbers as members?

   (c) How many subsets of S have exactly 4 elements?

   (d) How many four-element subsets of S contain exactly two odd numbers?

   (e) How many four-digit numbers can be made using the digits of S if a digit may be used repeatedly?

   (f) How many four-digit numbers can be made using the digits of S if a digit may be used only once?

   (g) How many even four-digit numbers bigger than 3000 can be made using the digits of S if a digit may be used only once?

9. (10 points) Suppose that if a, b, c, d and e are five different integers.

   (a) Must some pair of them differ by a multiple of 4? Explain your answer in detail.

   (b) Must any *four* different integers include a pair which differ by a multiple of 4?

10. (20 points) Let R be the following relation defined on the set $\{a, b, c, d\}$:

$$R = \{(a, a), (a, c), (a, d), (b, a), (b, b), (b, c), (b, d), (c, b), (c, c), (d, b), (d, d)\}.$$

Determine whether R is:

   (a) reflexive

   (b) symmetric

   (c) antisymmetric

   (d) transitive

11. (15 points) Suppose $A, B,$ and C are sets of integers such that $B \cap C = \phi, |B| = 15, |C| = 12, |A \cap B| = |A \cap C| = 2,$ and $|A \cup B| = 28$. Find each of the following:

    (a) $|A \cap \overline{C}|$

    (b) $|(A \cup B) \cap \overline{C}|$

    (c) $|A \cup B \cup C|$

12. (25 points) Let $A = \{1, 2, 3\}$.

    (a) Give an example of a 3×3 boolean matrix which represents a relation on A that is both symmetric and antisymmetric. Which entries of the matrix can be either 0 or 1?

    (b) Use the information in (a) to count the number of relations on A which are both symmetric and antisymmetric. (Remember that there are 2^9 relations on A).

    (c) How many relations on A are antisymmetric?

    (d) How many relations on A are symmetric?

    (e) How many relations on A are reflexive, symmetric, and transitive?

13. (20 points) Let $A = \{1, 2, 3, 4\}$. Find examples of relations on A which satisfy each of the following collections of conditions:

    (a) Reflexive and symmetric and not transitive.

    (b) Reflexive and antisymmetric and not transitive.

    (c) Symmetric and transitive and not antisymmetric.

    (d) Symmetric and not transitive and not reflexive.