**February 14, 2001**

Your name ________________________________

It is important that you **show your work**. There are 125 points available on this test.

1. (15 points) Find a pair of integers $m$ and $n$ such that $m/n$ is reduced and $m/n = 21.3\overline{64}$.

   **Solution:** Let $x = 21.3\overline{64}$ Compute $1000x - 10x$ to get $990x = 21151$. Divide by 990 to get $x = 21151/990$ and note that the numerator and denominator are relatively prime (ie, no common divisors bigger than 1).

2. (20 points)

   (a) Find the base 6 representation of 129.

   **Solution:** $333_6$

   (b) Find the base -6 representation of 129.

   **Solution:** $433_6$

   (c) Find the base 2 representation of 6.125.

   **Solution:** $110.001_2$

3. (20 points)

   (a) Use the division algorithm to find the unique integers $r$ and $q$ satisfying

   $$377 = 39q + r \text{ and } 0 \leq r < 39.$$

   **Solution:** $377 = 39 \cdot 9 + 26$

   (b) Solve the decanting problem for containers of sizes 377 and 39; that is find integers $x$ and $y$ satisfying

   $$377x + 39y = d$$

   where $d$ is the GCD of 39 and 377. containers of sizes 387 and 39; that is

   **Solution:** Repeated divisions followed by substitution results in $13 = 10 \cdot 39 - 1 \cdot 377$

4. (20 points) Notice that

$$1 = 1 = 1^2 \quad (1)$$
$$1 + 3 = 4 = 2^2 \quad (2)$$
$$1 + 3 + 5 = 9 = 3^2 \quad (3)$$
$$1 + 3 + 5 + 7 = 16 = 4^2 \quad (4)$$

(a) List the next three equations suggested by the pattern.
**Solution:**

$$1 + 3 + 5 + 7 + 9 = 25 = 5^2 \quad (5)$$
$$1 + 3 + 5 + 7 + 9 + 11 = 36 = 6^2 \quad (6)$$
$$1 + 3 + 5 + 7 + 9 + 11 + 13 = 49 = 7^2 \quad (7)$$

(b) Given that the four equations above are the $1^{st}, 2^{nd}, 3^{rd}$, and $4^{th}$, write the $n^{th}$ equation of the sequence. Notice that in the $4^{th}$ equation, the last summand is 7 (not 4).
**Solution:** $1 + 3 + 5 + 7 + 9 + \cdots + (2n - 1) = n^2$

(c) Use mathematical induction to prove that the $n^{th}$ equation is true for all positive integer values of $n$.
**Solution:** The base case for the proof is the first equation above, $1 = 1^2$. Given that $1 + 3 + 5 + 7 + 9 + \cdots + (2n - 1) = n^2$ holds for some $n \geq 1$, consider the left side of $P(n + 1)$. Note that $1 + 3 + 5 + 7 + 9 + \cdots + (2n - 1) + (2(n + 1) - 1) = n^2 + (2(n + 1) - 1) = n^2 + 2n + 1 = (n + 1)^2$. By PMI, the proposition $P(n)$ is true for all $n \geq 1$.

5. (15 points) Divisors Let $p, q$, and $r$ be three different prime numbers. In terms of $p, q$, and $r$, compute

   (a) $\text{GCD}(p^3q^2r, p^2qr^3)$
   **Solution:** $p^2qr$

   (b) $\text{LCM}(p^3q^2r, p^2qr^3)$
   **Solution:** $p^3q^2r^3$

   (c) the number of divisors of $p^3q^2r$.
   **Solution:** $(3 + 1)(2 + 1)(1 + 1) = 24$

6. (20 points) State the Fundamental Theorem of Arithmetic. Then use it to give an argument that the square root of 2 is irrational. Why is it not possible to prove that $\sqrt{4}$ is not rational using this method? Elaborate.

   **Solution:** FTA: Each integer $N > 1$ can be factored uniquely as a product of primes. Suppose $\sqrt{2}$ is rational. Then it can be expressed as $m/n$, for integers $m$ and $n$. Square both sides to get $2 = m^2/n^2$ or $2n^2 = m^2$. If $n = p_1^{e_1}p_2^{e_2}\cdots p_k^{e_k}$ and $m = q_1^{f_1}q_2^{f_2}\cdots q_l^{f_l}$. Thus $n^2 = p_1^{2e_1}p_2^{2e_2}\cdots p_k^{2e_k}$ and $m^2 = q_1^{2f_1}q_2^{2f_2}\cdots q_l^{2f_l}$. It follows that $2n^2$ factors into an odd number of primes and $m^2$ factors into an even number of primes. Therefore the two numbers $m^2$ and $2n^2$ cannot be equal.

7. (15 points) Prove that for any integer $n \geq 5$, $2^n > n^2$.

   **Solution:** The base case is simply $2^5 = 32 > 5^2 = 25$. Assume $P(n) : 2^n > n^2$, and consider the left side of $P(n+1)$. Note that $2^{n+1} = 2\cdot 2^n > 2\cdot n^2 > n^2+5n > n^2 + 3n > n^2 + 2n + 1 = (n + 1)^2$, so the inductive step is valid as well. By PMI, it follows that $P(n)$ is true for all $n \geq 5$.