

1. (15 points) Solve the decanting problem for containers of sizes 180 and 266; that is, find integers x and y satisfying

$$180x + 266y = d$$

where d is the GCD of 180 and 266.

Solution: Repeated division yields $2 = 8 - 6 = 8 - (86 - 10 \cdot 8) = 8 - 86 + 10 \cdot 8 = 11 \cdot 8 - 86 = 11(180 - 2 \cdot 86) - 86 = 11 \cdot 180 - 22 \cdot 86 - 86 = 11 \cdot 180 - 23(266 - 180) = 34 \cdot 180 - 23 \cdot 266$.

2. (12 points) Let $N = 17017$.

- (a) What is the prime factorization of N ?

Solution: $17017 = 7 \cdot 11 \cdot 13 \cdot 17$.

- (b) What is the remainder when N is divided by 9?

Solution: The sum of the digits is 16, so $N \equiv 16 \equiv 7 \pmod{9}$.

- (c) What is the remainder when N is divided by 11?

Solution: Since 11 is a factor of N , the remainder is 0.

- (d) How many (positive integer) divisors does N have?

Solution: Since N factors as a product of four distinct primes, it has $2 \cdot 2 \cdot 2 \cdot 2 = 16$ factors.

3. (10 points) Find a pair of relatively prime integers m and n such that $\frac{m}{n} = 0.\overline{857142}$.

Solution: Call the number x . Then $10^6x - x = 999999x = 857142$, so $x = 857142/999999$. This fraction reduces to $6/7$

4. (10 points) Find the base 5 representation of the number 717 in two ways.

Solution: By repeated subtraction, $717 = 625 + 92 = 625 + 3 \cdot 25 + 17 = 5^4 + 3 \cdot 5^2 + 3 \cdot 5 + 2 = 10332_5$. Also, by repeated division, we get the remainders 2, 3, 3, 0, and 1.

5. (10 points) Find the base 5 representation of the number $2/3$.

Solution: Repeated multiplication yields: $2/3 \cdot 5 = 10/3 = 3 + 1/3$. Then $1/3 \cdot 5 = 5/3 = 1 + 2/3$, and the repetition becomes clear. Thus $2/3 = 0.\overline{31}_5$.

6. (20 points) Notice that

$$1 = 1 \quad (1)$$

$$1 + 3 = 4 \quad (2)$$

$$1 + 3 + 5 = 9 \quad (3)$$

$$1 + 3 + 5 + 7 = 16 \quad (4)$$

(a) List the next three equations suggested by the pattern.

Solution:

$$1 + 3 + 5 + 7 + 9 = 25,$$

$$1 + 3 + 5 + 7 + 9 + 11 = 36,$$

$$1 + 3 + 5 + 7 + 9 + 11 + 13 = 49.$$

(b) Given that the four equations above are the 1st, 2nd, 3rd, and 4th, write the n^{th} equation of the sequence.

Solution:

$$1 + 3 + 5 + 7 + 9 + 11 + \cdots + (2n - 1) = n^2.$$

(c) Use mathematical induction to prove that the n^{th} equation is true for all positive integer values of n .

Solution: The base case for the proof is the first equation above, $1 = 1^2$. Given that $1 + 3 + 5 + 7 + 9 + \cdots + (2n - 1) = n^2$ holds for some $n \geq 1$, consider the left side of $P(n+1)$. Note that $1 + 3 + 5 + 7 + 9 + \cdots + (2n - 1) + (2(n+1) - 1) = n^2 + (2(n+1) - 1) = n^2 + 2n + 1 = (n+1)^2$. By PMI, the proposition $P(n)$ is true for all $n \geq 1$.

7. (10 points) You are playing a game of Bouton's Nim, and the position is $(17, 15, 13, 9, 6)$. Is this a safe position? If not, find the winning move.

Solution: The binary configuration is

$$\begin{array}{rcl} 6 & = & 110 \\ 9 & = & 1001 \\ 13 & = & 1101 \\ 15 & = & 1111 \\ 17 & = & 10001 \end{array}$$

so the three leftmost columns must be balanced. The only winning move is to take 4 counters from the pile of 17.

8. (8 points) What is the remainder when 3^{2002} is divided by 5?

Solution: Note that $3^2 \equiv -1 \pmod{5}$, so $3^{2002} = (3^2)^{1001} \equiv (-1)^{1001} \equiv -1 \equiv 4 \pmod{5}$. So the remainder is 4.

9. (10 points) Find digits a and b for which the number $11ab3$ is a multiple of 99.

Solution: The number must be both a multiple of 9 and of 11. Applying the divisibility tests for 9 and 11 gets A. either $5 + a + b = 9$ or $5 + a + b = 18$ and B. either $3 + a - b = 0$ or $3 + a - b = 11$. Only two of these are compatible and they lead to $a = 5$ and $b = 8$. Thus $11583 = 99 \cdot 117$ is a multiple of 99.

10. (15 points)

- (a) Suppose p, q , and r are different prime numbers. How many (positive integer) divisors does $N = p^2q^3r^5$ have?

Solution: The number of divisors is given by $D = (2+1)(3+1)(5+1) = 72$.

- (b) Find the greatest common divisor (GCD) of N and $M = pq^4$.

Solution: $GCD = p^1q^3r^0 = pq^3$.

- (c) Find the least common multiple (LCM) of N and $M = pq^4$.

Solution: $LCM = p^2q^4r^5$.

11. (10 points) Let $x = \log_2 5$. This means that $2^x = 5$. If x is rational, then there are integers m and n such that $2^{m/n} = 5$, or equivalently, $2^m = 5^n$. Explain in detail why $\log_2 5$ is irrational.

Solution: Because 2^m is even for any positive integer m , and 5^n is an odd number for any nonnegative integer n , $2^m \neq 5^n$ for any m, n .