Math 1165 **Test 1**

September 23, 2002 Your name ____________________________

1. (10 points) Find the base 6 representation of 2002.

   Solution: Repeated division produces 13134_6.

2. (10 points) Find the base 6 representation of 1/7.

   Solution: Repeated multiplication produces $0.\overline{05}_6$.

3. (10 points) Find a pair of relatively prime integers m and n for which $\frac{m}{n} = 1.2\overline{5}$. Two numbers are relatively prime if their greatest common divisor is 1.

   Solution: Let $x = 1.2\overline{5}$. Then $10x = 12.\overline{5}$. Subtract to get $x = \frac{11.3}{9}$. Thus $m = 113$ and $n = 90$.

4. (10 points) Find a base 6 digit d such that $22d1_6 = 6d4_9$.

   Solution: $22d1_6 = 432 + 72 + 6d + 1 = 505 + 6d$. On the other hand, $6d4_9 = 486 + 9d + 4 = 490 + 9d$. Therefore, $9d - 6d = 505 - 490 = 15$, which holds when $d = 5$.

5. (10 points) Find the best (winning) move in the game of Bouton's Nim $(14, 13, 12, 9)$.

   Solution: Take 6 from the pile with 14, or take 2 from the pile with 13 or take 2 from the pile with 12. The positions $(8, 13, 12, 9), (14, 11, 12, 9)$, and $(14, 13, 10, 9)$ are all balanced.

6. (12 points) Let $M = 123,123$ and let $N = 194,040$.

   (a) Find the prime factorizations of M and N.

   (b) Compute $LCM(M, N)$

   (c) Compute $GCD(M, N)$

   (d) Find the number of divisors of N.

   Solution: $N = 2^3 \cdot 3^2 \cdot 5 \cdot 7^2 \cdot 11$ and $M = 3 \cdot 7 \cdot 11 \cdot 13 \cdot 41$ so $LCM(M, N) = 2^3 \cdot 3^2 \cdot 5 \cdot 7^2 \cdot 11 \cdot 13 \cdot 41$ and $GCD(M, N) = 3 \cdot 7 \cdot 11$, and N has $(3 + 1)(2 + 1)(1 + 1)(2 + 1)(1 + 1) = 144$ divisors.

7. (15 points) Solve the decanting problem for containers of sizes 374 and 39; that is find integers x and y satisfying

$$374x + 39y = d$$

where d is the GCD of 39 and 374.

Solution: Repeated divisions followed by substitution results in $1 = 17 \cdot 374 - 163 \cdot 39$

8. (10 points) Find the representations of the integers 1 through 13 in base -4.

Solution: $1, 2, 3, 130, 131, 132, 133, 120, 121, 122, 123, 130, 131.$

9. (20 points) Look at the four equations below.

$$\begin{array}{lcl} 2 & = & 2 \cdot 1 \\ 2+4 & = & 3 \cdot 2 \\ 2+4+6 & = & 4 \cdot 3 \\ 2+4+6+8 & = & 5 \cdot 4 \end{array}$$

a. Write the next three equations in the sequence.

Solution:

$$2+4+6+8+10 = 6 \cdot 5$$

$$2+4+6+8+10+12 = 7 \cdot 6$$

$$2+4+6+8+10+12+14 = 8 \cdot 7$$

b. If the four equations above correspond to $k = 1, 2, 3,$ and 4, what is the n^{th} equation?

Solution:

$$P(n) : 2+4+6+8+\cdots+2n = (n+1)n$$

c. Prove by mathematical induction that the n^{th} equation is true for all integers $n \geq 1$.

Solution: First note that $P(1)$ is just the first equation above, $2 = 2 \cdot 1$. Next assume $P(n)$. Now the statement $P(n+1)$ is

$$2+4+6+8+\cdots+2n+2(n+1) = (n+1)(n+2).$$

The sum of all but the last term on the left is the right side of $P(n)$. Thus, $2+4+6+8+\cdots+2n+2(n+1) = (n+1)n+2(n+1) = (n+1)(n+2)$, by factoring. Thus, by the Principle of Mathematical Induction, $P(n)$ is true for all integers $n \geq 1$.