Math 1165 **Test 1**

February 17, 2003 Your name ____________________________

There are 122 points available on this test. You must show all your work.

1. (10 points) Find the base 8 representation of 2003.

   Solution: Repeated division produces 3723_8.

2. (10 points) Find the base 8 representation of 4/5.

   Solution: Repeated multiplication produces $0.\overline{6314}_8$.

3. (10 points) Find the base -8 representation of 175.

   Solution: Repeated division produces 337_{-8}.

4. (10 points) Find a pair of relatively prime integers m and n for which $\frac{m}{n} = 1.2\overline{25}$. Two numbers are relatively prime if their greatest common divisor is 1.

   Solution: Let $x = 1.2\overline{25}$. Then $100x = 12.\overline{25}$. Subtract to get $99x = 121.3$. Thus $m = 1213$ and $n = 990$.

5. (10 points) Find a digit d such that $d111_6 = 1d46_7$.

   Solution: $d111_6 = 216d + 36 + 6 + 1 = 216d + 43$. On the other hand, $1d46_7 = 343+49d+28+6 = 377+49d$. Therefore, $216d-49d = 377-43 = 334$, which holds when $d = 2$.

6. (12 points) Let $M = 324,324$ and let $N = 194,040$.

   (a) Find the prime factorizations of M and N.

   (b) Compute $LCM(M, N)$

   (c) Compute $GCD(M, N)$

   (d) Find the number of divisors of M.

   Solution: $N = 2^3 \cdot 3^2 \cdot 5 \cdot 7^2 \cdot 11 \cdot 13$ and $M = 2^2 \cdot 3^4 \cdot 7 \cdot 11 \cdot 13$ so $LCM(M, N) = 2^3 \cdot 3^4 \cdot 5 \cdot 7^2 \cdot 11 \cdot 13 = 22702680$ and $GCD(M, N) = 2^2 \cdot 3^4 \cdot 7 \cdot 11 = 2772$, and M has $(2 + 1)(4 + 1)(1 + 1)(1 + 1)(1 + 1) = 120$ divisors.

7. (15 points) Solve the decanting problem for containers of sizes 99 and 79; that is find integers x and y satisfying

$$99x + 79y = d$$

where d is the GCD of 99 and 79.

Solution: Repeated divisions followed by substitution results in $1 = 20 - 19 = 20 - (79 - 3 \cdot 20) = 4 \cdot 20 - 79 = 4(99 - 79) - 79 = 4 \cdot 99 - 5 \cdot 79$, so $x = 4$ and $y = -5$.

8. (15 points) Find the remainder in each case below.

(a) When $N = 5^{2003}$ is divided by 7.

Solution: We use the notation of modular arithmetic. Note that $5 \equiv 5 (\text{mod } 7)$, $5^2 \equiv 4 (\text{mod } 7)$, $5^3 \equiv 6 (\text{mod } 7)$, $5^4 \equiv 2 (\text{mod } 7)$, $5^5 \equiv 3 (\text{mod } 7)$, and $5^6 \equiv 1 (\text{mod } 7)$. Therefore $5^{6n} \equiv 1 (\text{mod } 7)$ for any integer $n > 0$, and $5^{2003} = 5^{1998} \cdot 5^5 \equiv 1^{1998} \cdot 5^5 \equiv 1 \cdot 3 \equiv 3 (\text{mod } 7)$.

(b) When $N = 1234445556667778889999$ is divided by 9.

Solution: The sum of the digits of N is 123, which is congruent to 6 modulo 9. Therefore $N \equiv 6 (\text{mod } 9)$.

(c) When $N = 1234445556667778889999$ is divided by 11.

Solution: The alternating sum of the digits of N is $64 - 59 = 5$, which is congruent to 5 modulo 11. Therefore $N \equiv 5 (\text{mod } 11)$.

9. (10 points) Find the best (winning) move in the game of Bouton's Nim $(27, 23, 22, 19)$.

   Solution: Take 9 from the pile with 27. The position $(18, 23, 22, 19)$ the only balanced position available.

10. (20 points) Look at the four equations below.

$$\begin{aligned}
1 \cdot 1! &= 2! - 1 \\
1 \cdot 1! + 2 \cdot 2! &= 3! - 1 \\
1 \cdot 1! + 2 \cdot 2! + 3 \cdot 3! &= 4! - 1 \\
1 \cdot 1! + 2 \cdot 2! + 3 \cdot 3! + 4 \cdot 4! &= 5! - 1
\end{aligned}$$

   (a) Write the next three equations in the sequence.

   Solution:

$$\begin{aligned}
1 \cdot 1! + 2 \cdot 2! + 3 \cdot 3! + 4 \cdot 4! + 5 \cdot 5! &= 6! - 1 \\
1 \cdot 1! + 2 \cdot 2! + 3 \cdot 3! + 4 \cdot 4! + 5 \cdot 5! + 6 \cdot 6! &= 7! - 1 \\
1 \cdot 1! + 2 \cdot 2! + 3 \cdot 3! + 4 \cdot 4! + 5 \cdot 5! + 6 \cdot 6! + 7 \cdot 7! &= 8! - 1
\end{aligned}$$

   (b) If the four equations above correspond to $k = 1, 2, 3,$ and 4, what is the n^{th} equation?

   Solution: $1 \cdot 1! + 2 \cdot 2! + 3 \cdot 3! + \cdots + +k \cdot k! = (k + 1)! - 1$.

   (c) Prove by mathematical induction that the n^{th} equation is true for all integers $n \geq 1$.

   Solution: Suppose we have a positive integer k for which $P(k)$ is true. Then, $1 \cdot 1! + 2 \cdot 2! + 3 \cdot 3! + \cdots + +k \cdot k! = (k+1)! - 1$. Now to prove $P(k+1)$, examine the left side of $P(k+1)$. Thus $1 \cdot 1! + 2 \cdot 2! + 3 \cdot 3! + \cdots + +k \cdot k! + (k+1)(k+1)! = (k+1)! - 1 + (k+1)(k+1)! = (k+1)![1+k+1] - 1 = (k+2)! - 1$. This completes the inductive step. The base case is the first equation above. Therefore, by the Principle of Mathematical Induction, $P(n)$ is true for all positive integers n.