

March 27, 2000

Your name _____

It is important that you **show your work**. The total value of this test is 108 points.

1. (16 points) Let $S = \{1, 2, 3, 4, 5, 6, 7\}$.
 - a. How many non-empty subsets does S have?
 - b. How many non-empty subsets of S consist entirely of odd numbers?
 - c. How many subsets of S have exactly 4 elements?
 - d. How many four element subsets of S contain exactly two odd numbers?

2. (8 points) Suppose A, B , and C are finite sets of numbers such that $A \cap B = \phi$, $B \cap C = \phi$, $|A| = 13$, $|B| = 15$, $|C| = 12$, $|A \cap C| = 2$. Furthermore, $|A \cup B| = 28$. Find each of the following:
 - (a) $|A \cap \overline{C}|$
 - (b) $|A \cup B \cup C|$

3. (15 points) Let $\mathcal{U} = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$. Let $A = \{1, 2, 3, 4, 5\}$, $B = \{4, 5, 6, 7\}$, and $C = \{1, 5, 8\}$. Recall that \times denotes Cartesian product and \overline{X} denotes the complement of X with respect to \mathcal{U} . Find each of the following. Recall that $A \oplus B$ denotes the symmetric difference of A and B .

(a) $|(A \cap \overline{B}) \times (\overline{A} \cap C)|$

(b) $|(A \oplus B) \cup C|$

(c) $|(A \times A) \cup (A \times B) \cup (A \times C)|$

4. (12 points) Let A and B be sets. Let f be a function from A to B . What is meant by

(a) f is a one-to-one function.

(b) f is an onto function.

(c) f is a bijective function.

5. (12 points) Reproduce the proof that $[0, 1] \approx [0, 1] \times [0, 1]$.
6. (15 points) Define what is meant by the characteristic function of a set A , where \mathcal{U} denotes the universal set. Recall that for any sets A and B , $f_{A \cup B} = f_A + f_B - f_A f_B$ and $f_{A \cap B} = f_A \cdot f_B$. Use characteristic functions and the two identities to prove that \cap distributes over \cup ; that is, for any sets A, B , and C ,

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C).$$

7. (20 points) Let $S = \{1, 2, 3, 4, 5, 6, 7, 8\}$.
- (a) How many four-digit numbers can be made using the digits of S , assuming repetition of digits is not allowed?
 - (b) How many four-digit numbers can be made using the digits of S , assuming repetition of digits is allowed?
 - (c) How many four-digit numbers can be made using the digits of S , assuming the digits must be in increasing order from left to right? For example 1345 counts, but 1156 does not.
 - (d) How many even four-digit numbers can be made using the digits of S , assuming repetition of digits is not allowed?

8. (10 points) Give either an algebraic or a combinatorial proof of the identity

$$C_r^n + C_{r+1}^n = C_{r+1}^{n+1}.$$