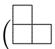
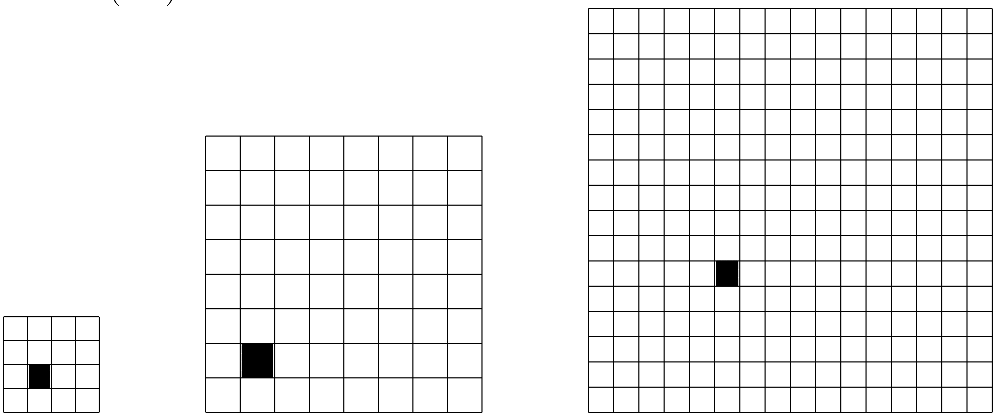
Math 1165 **Discrete Math** **Test 2.**

April 4, 2001

Your name ______________________________

It is important that you **show your work**. There are 134 points available on this test.

1. (10 points) Show how to tile the punctured chess boards below with L-shaped triominoes ().



2. (10 points) Four sets A, B, C, and D are given. Each has cardinality 100. The cardinality of the intersection of any two of them is 50, and the cardinality of the intersection of any three is 25. Finally, $|A \cap B \cap C \cap D| = 5$. What is $|A \cup B \cup C \cup D|$?

3. (10 points) In a group of 100 students, the following facts are known:

   - 50 take math,
   - 40 take computing,
   - 35 take chemistry,
   - 12 take both math and computing,
   - 10 take math and chemistry,
   - 11 take chemistry and computing, and
   - 5 take all three subjects.

   How many take none of the three subjects?

4. (10 points) You have a drawer full of socks. There are 6 red, 8 blue, 10 black, and 12 brown. How many socks must be removed from the drawer (in the dark) to be guaranteed that two of the socks removed match in color? Explain how the pigeonhole principle applies in this problem.

5. (16 points) Recursion

   (a) Consider the sequence $1, 1/2, 1/3, 1/4, \ldots$.
      i. Find a closed form formula.
      ii. Find a first order recursive definition of the sequence.

   (b) Consider the sequence $1, 1/2, 1/4, 1/8, \ldots$.
      i. Find a closed form formula.
      ii. Find a first order recursive definition of the sequence.

   (c) Consider the sequence $1/3, 1/5, 1/9, 1/17, 1/33, \ldots$.
      i. Find a closed form formula.
      ii. *Find a recursive definition of the sequence.

   (d) Consider the sequence $0, 5, 8, 17, 24, 37, 48, 65$.
      i. Find a closed form formula.
      ii. *Find a recursive definition of the sequence.

6. (15 points) Characteristic Functions. Recall that the characteristic function f_A of a set A is given by

$$f_A(x) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{otherwise} \end{cases}$$

   (a) If A and B are given sets with characteristic functions f_A and f_B, describe each of the following in terms of f_A and f_B.

i. $f_{\overline{A}}$

ii. $f_{A\cap B}$

iii. $f_{A\cup B}$

(b) Use characteristic functions to prove that intersection distributes over union; that is, if A, B, and C are any three sets, then $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.

7. (12 points) Prove that from every five-element set of natural numbers it is possible to select three whose sum is a multiple of 3. Hint: use modular arithmetic.

8. (15 points) Consider the recurrence relation $s_n = 2s_{n-1} + 3s_{n-2}$ with initial values $s_0 = 0$ and $s_1 = 8$.

   (a) Find the characteristic equation.

   (b) Find the two roots λ_1, λ_2 of the equation.

   (c) The general solution is given by

   $$s_n = c_1\lambda_1^n + c_2\lambda_2^n.$$

   Use your values of λ_1 and λ_2 to find c_1 and c_2 satisfying the initial conditions.

   (d) Compute the value of s_{100}.

9. (12 points) Equivalence of sets. Let $N = \{1, 2, 3, 4, \ldots$ the natural numbers, and let $S = \{1, 4, 9, 16, \ldots$ be the perfect squares. Show that $N \sim S$ by finding a bijection f from N to S. (3 points for the definition of f, 9 points for the proof that it is a bijection)

10. (12 points) Recall the function $f : [0, 1] \to [0, 1] \times [0, 1]$ defined by

$$f(0.x_1x_2x_3\ldots) = (0.x_1x_3x_5\ldots, 0.x_2x_4x_6\ldots).$$

Compute each of the following. Leave your answer in the form in which the information is given (ie. fraction/fraction; decimal/decimal).

   (a) $f(4/33)$

   (b) $f(0.\overline{123})$

   (c) find x if $f(x) = (1/9, 1/3)$

11. (12 points) Prove that for any positive integer n,

$$3^0 + 3 + 3^2 + \cdots + 3^n = \frac{3^{n+1} - 1}{2}.$$