
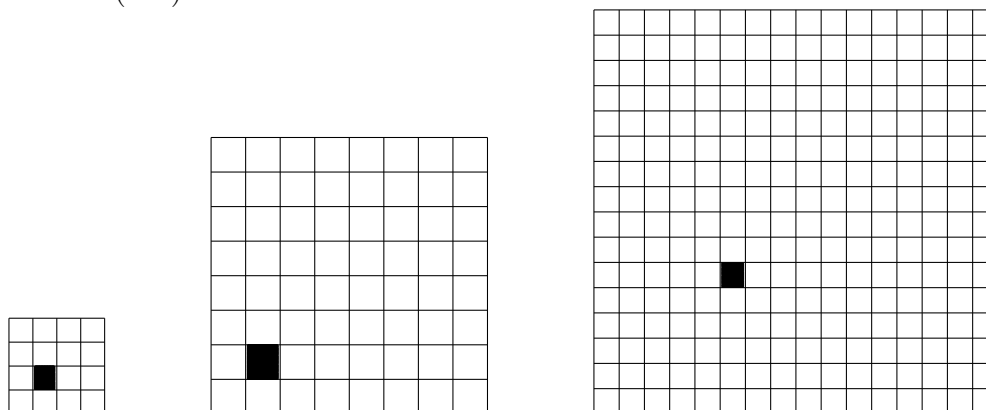


April 4, 2001

Your name _____

It is important that you **show your work**. There are 134 points available on this test.

1. (10 points) Show how to tile the punctured chess boards below with L-shaped triominoes ().



Solution: The point of this problem is to see if you understand how the solution for the 4×4 board shows up in the solutions for the larger boards. The induction argument we saw early in the course applies. Divide the $2^n \times 2^n$ board into four quadrants, one of which has the deleted square. Tile that quadrant, and then place a triomino around the corner at the center of the board so that effectively deletes one square from each of the other three quadrants.

2. (10 points) Four sets A, B, C , and D are given. Each has cardinality 100. The cardinality of the intersection of any two of them is 50, and the cardinality of the intersection of any three is 25. Finally, $|A \cap B \cap C \cap D| = 5$. What is $|A \cup B \cup C \cup D|$?

Solution: The inclusion-exclusion principle applies to give $|A \cup B \cup C \cup D| = 100 + 100 + 100 + 100 - (50 + 50 + 50 + 50 + 50 + 50) + (25 + 25 + 25 + 25) - 5 = 195$.

3. (10 points) In a group of 100 students, the following facts are known:

- 50 take math,
- 40 take computing,
- 35 take chemistry,
- 12 take both math and computing,
- 10 take math and chemistry,
- 11 take chemistry and computing, and
- 5 take all three subjects.

How many take none of the three subjects?

Solution: Another inclusion-exclusion problem. Let M, C , and Ch denote the sets of students who study math, computing, and chemistry respectively. Then $|A \cup C \cup Ch| = |A| + |C| + |Ch| - |AC| - |ACh| - |CCh| + |ACCh| = (50 + 40 + 35) - (12 + 10 + 11) + 5 = 97$, which leaves 3 students studying none of the three subjects.

4. (10 points) You have a drawer full of socks. There are 6 red, 8 blue, 10 black, and 12 brown. How many socks must be removed from the drawer (in the dark) to be guaranteed that two of the socks removed match in color? Explain how the pigeonhole principle applies in this problem.

Solution: Five socks is enough. Consider four pigeonholes, one for each color. Then five socks distributed among four pigeonholes is enough to guarantee that some pigeonhole contains at least 2 socks, and these two form a matching pair.

5. (16 points) Recursion

(a) Consider the sequence $1, 1/2, 1/3, 1/4, \dots$

i. Find a closed form formula.

Solution: $a_n = 1/n, n = 1, 2, \dots$

ii. Find a first order recursive definition of the sequence.

Solution: $a_1 = 1, a_{n+1} = (a_n^{-1} + 1)^{-1}$

(b) Consider the sequence $1, 1/2, 1/4, 1/8, \dots$

i. Find a closed form formula.

Solution: $a_n = 1/2^n, n = 0, 1, 2, \dots$

ii. Find a first order recursive definition of the sequence.

Solution: $a_1 = 1, a_{n+1} = a_n/2$

(c) Consider the sequence $1/3, 1/5, 1/9, 1/17, 1/33, \dots$

i. Find a closed form formula.

Solution: $a_n = 1/(1 + 2^n)$.

ii. *Find a recursive definition of the sequence.

Solution: $a_1 = 1/3, a_{n+1} = (1 + 2(a_n^{-1} - 1))^{-1}$

(d) Consider the sequence 0, 5, 8, 17, 24, 37, 48, 65.

i. Find a closed form formula.

Solution: $a_n = n^2 + (-1)^n$.

ii. *Find a recursive definition of the sequence.

Solution: $a_1 = 0, a_{n+1} = (\sqrt{a_n - (-1)^n} + 1)^2 + (-1)^{n+1}$ or $a_1 = 0, a_{n+1} = a_n + 2n - 1 + 2(-1)^{n+1}$

6. (15 points) Characteristic Functions. Recall that the characteristic function f_A of a set A is given by

$$f_A(x) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{otherwise} \end{cases}$$

(a) If A and B are given sets with characteristic functions f_A and f_B , describe each of the following in terms of f_A and f_B .

i. $f_{\bar{A}}$

Solution: $f_{\bar{A}} = 1 - f_A$

ii. $f_{A \cap B}$

Solution: $f_{A \cap B} = f_A \cdot f_B$

iii. $f_{A \cup B}$

Solution: $f_{A \cup B} = f_A + f_B - f_A \cdot f_B$

(b) Use characteristic functions to prove that intersection distributes over union; that is, if A, B , and C are any three sets, then $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.

Solution: To prove that $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$, it suffices to prove that $f_{A \cap (B \cup C)} = f_{(A \cap B) \cup (A \cap C)}$. To that end, note that $f_{A \cap (B \cup C)} = f_A \cdot f_{B \cup C} = f_A(f_B + f_C - f_B \cdot f_C) = f_A \cdot f_B + f_A \cdot f_C - f_A f_B f_C$, which is the same as $f_{(A \cap B) \cup (A \cap C)}$.

7. (12 points) Prove that from every five-element set of natural numbers it is possible to select three whose sum is a multiple of 3. Hint: use modular arithmetic.

Solution: Put each of the five numbers (pigeons) into a hole by its remainder upon division by 3. There are 3 possible remainders. Suppose that some hole is empty. Then by the extended PhP, one hole must contain at least 3 integers, and the sum of these is a multiple of 3. On the other hand if each hole has at least one integer, then take one from each pigeonhole. Their sum is a multiple of 3. OR

Solution: Distribute the five integers into three pigeonholes according to their remainder when divided by 3. The possible remainders are 0, 1 and 2. If any of these remainders occurs three times, then those three integers have a sum that is a multiple of 3. If no pigeonhole contains three integers, then each pigeonhole must have at least one integer. Pick one number from each of these. Their sum is $x + y + z \equiv 0 + 1 + 2 \equiv 0 \pmod{3}$.

8. (15 points) Consider the recurrence relation $s_n = 2s_{n-1} + 3s_{n-2}$ with initial values $s_0 = 0$ and $s_1 = 8$.

- (a) Find the characteristic equation.

Solution: Assume there is a solution of the form $s_n = \lambda^n$. Then $\lambda^n - 2\lambda^{n-1} - 3\lambda^{n-2} = 0$. Factor out λ^{n-2} to get $\lambda^{n-2}(\lambda^2 - 2\lambda - 3) = 0$.

- (b) Find the two roots λ_1, λ_2 of the equation.

Solution: Factor still more to get $(\lambda + 1)(\lambda - 3) = 0$. Hence there are two roots, $\lambda_1 = -1$ and $\lambda_2 = 3$.

- (c) The general solution is given by

$$s_n = c_1\lambda_1^n + c_2\lambda_2^n.$$

Use your values of λ_1 and λ_2 to find c_1 and c_2 satisfying the initial conditions.

Solution: The values obtained are $c_1 = -2$ and $c_2 = 2$ and the unique solution is $s_n = -2(-1)^n + 2 \cdot 3^n$.

- (d) Compute the value of s_{100} .

Solution: $s_{100} = 2 \cdot 3^{100} - 2$.

9. (12 points) Equivalence of sets. Let $N = \{1, 2, 3, 4, \dots\}$ the natural numbers, and let $S = \{1, 4, 9, 16, \dots\}$ be the perfect squares. Show that $N \sim S$ by finding a bijection f from N to S . (3 points for the definition of f , 9 points for the proof that it is a bijection)

Solution: Define $f : N \rightarrow S$ as follows: $f(n) = n^2$. To see that f is 1-to-1, suppose $a^2 = f(a) = f(a') = a'^2$. Then we can take positive square roots of

both sides to see that $a = a'$. To see that f is onto, suppose $b \in S$. Then $\sqrt{b} \in N$, and $f(\sqrt{b}) = b$.

10. (12 points) Recall the function $f : [0, 1] \rightarrow [0, 1] \times [0, 1]$ defined by

$$f(0.x_1x_2x_3\dots) = (0.x_1x_3x_5\dots, 0.x_2x_4x_6\dots).$$

Compute each of the following. Leave your answer in the form in which the information is given (ie. fraction/fraction; decimal/decimal).

(a) $f(4/33)$

Solution: $f(4/33) = f(0.\overline{12}) = (0.\overline{1}, 0.\overline{2}) = (1/9, 2/9)$

(b) $f(0.\overline{123})$

Solution: $f(0.\overline{123}) = (0.\overline{132}, 0.\overline{213})$

(c) find x if $f(x) = (1/9, 1/3)$

Solution: $(1/9, 1/3) = (0.\overline{1}, 0.\overline{3}) = f(0.\overline{13}) = f(13/99)$

11. (12 points) Prove that for any positive integer n ,

$$3^0 + 3 + 3^2 + \dots + 3^n = \frac{3^{n+1} - 1}{2}.$$

Solution: The inductive step is given by

$$\begin{aligned} 3^0 + 3 + 3^2 + \dots + 3^n + 3^{n+1} &= \frac{3^{n+1} - 1}{2} + 3^{n+1} \\ &= \frac{3^{n+1} - 1}{2} + \frac{2 \cdot 3^{n+1}}{2} \\ &= \frac{3^{n+1} - 1 + 2 \cdot 3^{n+1}}{2} \\ &= \frac{3 \cdot 3^{n+1} - 1}{2} \\ &= \frac{3^{n+2} - 1}{2}. \end{aligned}$$