Math 1165 **Test 2**

June 21, 2001

Your name ______________________________

It is important that you **show your work**. The total value of this test is 165 points.

1. (20 points) Let A and B be sets with characteristic functions f_A and f_B. Compute the following characteristic functions of the terms of f_A and f_B.

   (a) $f_{A \cap B}$

   (b) $f_{A \cup B}$

   (c) $f_{\overline{A}}$

   (d) Prove the *DeMorgan* property $\overline{A \cup B} = \overline{A} \cap \overline{B}$ using the characteristic functions above.

2. (10 points) Suppose $f(n) = 5f(\frac{n}{2}) + 2n - 1$ and $f(4) = 40$. Find $f(1)$.

3. (10 points) Find the number of integers from 1 to 300 inclusive that are divisible by neither 7 nor 8.

4. (12 points) Prove that there is no function from $Z^+ = \{1, 2, 3, \ldots\}$ **ONTO** $[0, 1]$. That is, there is no function from the positive integers onto the closed unit interval.

5. (30 points) The counting of sets of sums. For each case below list five members of the set you're trying to count, and find the cardinality of the set.

   (a) How many integers can be expressed as a sum of two or more different members of the set $S = \{0, 1, 2, 3, 7, 14\}$?

   (b) How many integers can be expressed as the sum of exactly five distinct members of the set $T = \{1, 2, 4, 8, 16, 32, 64\}$?

   (c) John has 1 penny, 2 nickels, 3 dimes, 4 quarters, and 1 dollar. For how many different amounts can John make an exact purchase (with no change required)?

   (d) How many numbers can be expressed as a sum of four distinct members of the set $\{17, 21, 25, 29, 33, 37, 41, 44\}$? Hint: Notice that all but one of the numbers is congruent to 1 (mod 4).

   (e) Suppose you live in a country that has the system of coins with values $1, 3, 9, 27,$ and 81, and suppose you have two coins of each type. How many different amounts can you make with your 10 coins?

6. (20 points) Let $S = \{1, 2, \ldots, 11\}$. For each property below find the number of subsets T of S that have the property.

   (a) The number of odd integer members and the number of even integer members of T is the same.

   (b) T has two even members and one odd member.

   (c) T is not empty and consists only of prime numbers.

   (d) T has three elements whose sum is at most 10.

   (e) T has at least two even members and at most three odd members.

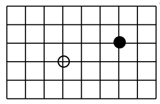
7. (8 points) Suppose that the characteristic equation of a linear homogeneous recurrence relation with constant coefficients is

$$(\lambda - 3)^4(\lambda - 2)^3(\lambda + 6) = 0.$$

   Write the general solution of the recurrence relation.

8. (15 points) Imagine that the 5×8 grid below represents the streets of a part of the city where you live. You must walk 13 blocks to get from the lower left corner at A to the upper right corner at B.

B



A

(a) How many different 13-block walks are there?

(b) How many 13-block walks go through both the $\bullet$ corner and the $\circ$ corner.

(c) How many 13-block walks avoid the terrible corner marked with the bullet $\bullet$?

(d) How many different 14-block walks are there from A to B?

9. (10 points) Show that the set $A = \{2, 4, 6, 8, \ldots\}$ of positive even integers is equivalent (in the sense of Cantor) to the set Z of all integers. The important part of this problem is to define the bijection between the two sets and to show that it is both 1-1 and onto.

10. (10 points) Solve: $a_n = -7a_{n-1} - 10a_{n-2}$ for $n \geq 2$ with the initial values $a_0 = 3$, $a_1 = 3$.

11. (20 points) Suppose $\mathcal{U}$ is a 12 element set with subsets A, B, and C satisfying $|A| = |B| = |C| = 6$, $|AB| = |BC| = 3$, $|AC| = 2$, and $|ABC| = 1$. Compute the following cardinalities.

    (a) $|A \cup (B \cap \overline{C})|$

    (b) $|(A \times A) \cup (B \times B) \cup (C \times C)|$

    (c) $|(A \cap B) \times (B \cap C) \times (A \cap C)|$

    (d) $|\overline{(A \cup B \cup C) \times (A \cup B \cup C)}|$

    (e) $|\overline{(A \cup B \cup C)} \times \overline{(A \cup B \cup C)}|$