Math 1165 **Test 2**

June 21, 2001

Your name ______________________________

It is important that you **show your work**. The total value of this test is 165 points.

1. (20 points) Let A and B be sets with characteristic functions f_A and f_B. Compute the following characteristic functions of the terms of f_A and f_B.

   (a) $f_{A\cap B}$
   Solution: $f_{A\cap B} = f_A \cdot f_B$

   (b) $f_{A\cup B}$
   Solution: $f_{A\cup B} = f_A + f_B - f_A f_B$

   (c) $f_{\overline{A}}$
   Solution: $f_{\overline{A}} = 1 - f_A$

   (d) Prove the *DeMorgan* property $\overline{A \cup B} = \overline{A} \cap \overline{B}$ using the characteristic functions above.

   Solution:

$$\begin{aligned} f_{\overline{A\cup B}} &= 1 - f_{A\cup B} \\ &= 1 - (f_A + f_B - f_A f_B) \\ &= 1 - f_A - f_B + f_A f_B) \\ &= (1 - f_A)(1 - f_B) \\ &= f_{\overline{A}} f_{\overline{B}} \\ &= f_{\overline{A} \quad \overline{B}} \end{aligned}$$

2. (10 points) Suppose $f(n) = 5f(\frac{n}{2}) + 2n - 1$ and $f(4) = 40$. Find $f(1)$.

   Solution: First use $f(4)$ to find $f(2)$: $f(4) = 5f(\frac{4}{2}) + 2 \cdot 4 - 1$. Therefore $40 = 5f(2) + 7$, or $f(2) = \frac{33}{5}$.

   Then use $f(2)$ to find $f(1)$: $f(2) = 5f(\frac{2}{2}) + 2 \cdot 2 - 1$.

   Therefore $\frac{33}{5} = 5f(1) + 3$, or $f(1) = \frac{18}{25}$.

3. (10 points) Find the number of integers from 1 to 300 inclusive that are divisible by neither 7 nor 8.

   Solution: Let A be the integers from 1 to 300 divisible by 7 and B be the set of integers from 1 to 300 divisible by 8. The number of positive integers less than or equal to 300 that are divisible by neither 7 nor 8 is $\left|\overline{A} \cap \overline{B}\right| = \left|\overline{A \cup B}\right| = 300 - |A \cup B| = 300 - \left(\left\lfloor \frac{300}{7} \right\rfloor + \left\lfloor \frac{300}{8} \right\rfloor - \left\lfloor \frac{300}{56} \right\rfloor\right) = 300 - 42 - 37 + 5 = 226.\}$

4. (12 points) Prove that there is no function from $Z^+ = \{1, 2, 3, \ldots\}$ **ONTO** $[0, 1]$. That is, there is no function from the positive integers onto the closed unit interval.

   Solution: This is Cantor's diagonalization method. Suppose $f : Z^+ \to [0, 1]$. Write the binary representation of $f(1), f(2), \ldots,$ in a matrix as shown:

$$\begin{aligned} f(1) &= x_{1,1}x_{1,2}x_{1,3}\ldots \\ f(2) &= x_{2,1}x_{2,2}x_{2,3}\ldots \\ f(3) &= x_{3,1}x_{3,2}x_{3,3}\ldots \\ &\vdots \end{aligned}$$

   Next construct a number y in $[0, 1]$ that is different from any of the $f(n)$ as follows; $y = y_1y_2y_3\ldots$ where $y_i = x_{i,i} + 1$, with the understanding that $y_i = 0$ if $x_{i,i} = 9$. Comparing each $f(n)$ with y we can see that they are different in the n^{th} position, so they are different numbers. Thus y is not $f(n)$ for any value of n. This contradiction shows that f cannot map Z^+ onto $[0.1]$

5. (30 points) The counting of sets of sums. For each case below list five members of the set you're trying to count, and find the cardinality of the set.

   (a) How many integers can be expressed as a sum of two or more different members of the set $S = \{0, 1, 2, 3, 7, 14\}$?

   Solution: Just list the numbers that can be represented:$1 = 0 + 1, 2 = 0 + 2, 3, 4, 5, 6, 7, 8, 9, \ldots, 27$. The sum of all the elements of the set is 27, so no number larger than 27 can be represented.

   (b) How many integers can be expressed as the sum of exactly five distinct members of the set $T = \{1, 2, 4, 8, 16, 32, 64\}$?

   Solution: For each binary sever-tuple with exactly five 1's, there is a unique sum. Therefore there are $\binom{7}{5} = 21$ sums.

   (c) John has 1 penny, 2 nickels, 3 dimes, 4 quarters, and 1 dollar. For how many different amounts can John make an exact purchase (with no change required)?

   Solution: Every value of the form $x.yz$ where $x = 0$ or 1, $y =$ any digit, and $z = 0, 1, 5$ or 6 can be achieved. Also, all values of the form $2.yz$ where $y = 0, 1, 2, 3$ and $z = 0, 1, 5$, or 6; and 2.40 and 2.41. There are $79 = 80 - |\{0.00\}| = 79$ of the first type and 16 of the second. Therefore, there are $79 + 16 + 2 = 97$ altogether.

   (d) How many numbers can be expressed as a sum of four distinct members of the set $\{17, 21, 25, 29, 33, 37, 41, 44\}$? Hint: Notice that all but one of the numbers is congruent to 1 (mod 4).

   Solution: Count those sums that include the 44 and those that don't. There are $140/4 - 92/4 + 1 = 13$ or the latter and $(155 + 1)/4 - (107 + 1)/4 + 1 = 13$ of the latter for a total of 26 numbers.

   (e) Suppose you live in a country that has the system of coins with values $1, 3, 9, 27,$ and 81, and suppose you have two coins of each type. How many different amounts can you make with your 10 coins?

   Solution: Think ternary. All the numbers from 1 up to $3^5 - 1 = 242$, which is 242 values.

6. (20 points) Let $S = \{1, 2, \ldots, 11\}$. For each property below find the number of subsets T of S that have the property.

   (a) The number of odd integer members and the number of even integer members of T is the same.

   Solution: A few sets with this property are $\phi, \{1, 2\}, \{1, 2, 5, 6\}$. There are $\binom{5}{0}\binom{6}{0} + \binom{5}{1}\binom{6}{1} + \binom{5}{2}\binom{6}{2} + \binom{5}{3}\binom{6}{3} + \binom{5}{4}\binom{6}{4} + \binom{5}{5}\binom{6}{5} = 1 \cdot 1 + 5 \cdot 6 + 10 \cdot 15 + 10 \cdot 20 + 5 \cdot 15 + 1 \cdot 6 = 1 + 30 + 150 + 200 + 75 + 6 = 462.$

   (b) T has two even members and one odd member.

   Solution: $\binom{5}{2}\binom{6}{1} = 10 \cdot 6 = 60.$

   (c) T is not empty and consists only of prime numbers.

   Solution: All the non-empty subsets of $\{2, 3, 5, 7, 11\}$ qualify, and there are $2^5 - 1 = 31$ such subsets

   (d) T has three elements whose sum is at most 10.

   Solution: A brute force search shows that only 11 sets have this property.

   (e) T has at least two even members and at most three odd members.

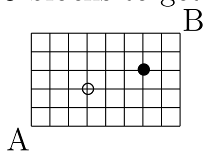
   Solution: Construct T by constructing the subset E of even member and the subset O of odd members. E has either $2, 3, 4$, or 5 members and O has either $0, 1, 2$, or 3 members. Thus, there are $\binom{5}{2} + \binom{5}{3} + \binom{5}{4} + \binom{5}{5} = 26$ choices for E, and $\binom{6}{0} + \binom{6}{1} + \binom{6}{2} + \binom{6}{3} = 42$ choices for O. So there are $26 \cdot 42 = 1092$ subsets T that satisfy the conditions.

7. (8 points) Suppose that the characteristic equation of a linear homogeneous recurrence relation with constant coefficients is

$$(\lambda - 3)^4(\lambda - 2)^3(\lambda + 6) = 0.$$

   Write the general solution of the recurrence relation.

   Solution: $a_n = a3^n + bn3^n + cn^2 3^n + dn^3 3^n + e2^n + fn2^n + gn^2 2^n + h(-6)^n.$

8. (15 points) Imagine that the 5×8 grid below represents the streets of a part of the city where you live. You must walk 13 blocks to get from the lower left corner at A to the upper right corner at B. 

   (a) How many different 13-block walks are there?

   Solution: Each path can be coded as an 13 letter string, each letter of which is an u(for up) or an r(for right). There are $\binom{13}{5} = 1287$ such strings.

   (b) How many 13-block walks go through both the $\bullet$ corner and the $\circ$ corner.

   Solution: There are $\binom{5}{2}\binom{4}{1}\binom{4}{2} = 10 \cdot 4 \cdot 6 = 240$ ways to go through both special corners.

   (c) How many 13-block walks avoid the terrible corner marked with the bullet $\bullet$?

   Solution: Notice that there are $\binom{9}{3}\binom{4}{2} = 84 \cdot 6 = 504$ ways to go through the terrible $\bullet$ corner, so the must be $1287 - 504 = 783$ ways to avoid it.

   (d) How many different 14-block walks are there from A to B?

   Solution: There are none because each even unit walk must end on a vertex both of whose coordinates are even or both odd.

9. (10 points) Show that the set $A = \{2, 4, 6, 8, \ldots\}$ of positive even integers is equivalent (in the sense of Cantor) to the set Z of all integers. The important part of this problem is to define the bijection between the two sets and to show that it is both 1-1 and onto.

   Solution: Define a function $f : Z \to A$ by

$$f(x) = \begin{cases} 4n + 2 & \text{if } n \geq 0 \\ -4n & \text{if } n < 0 \end{cases}$$

   Thus $f(0) = 2, f(-1) = 4, f(1) = 6, f(-2) = 8$ and $f(2) = 10$. Clearly $f(n) \in A$ for each $n \in Z$. To see that f is one-to-one, suppose $m < n$ are integers. We need to show that $f(m) \neq f(n)$. If both m and n are nonnegative, then $f(m) = 4m+2 < 4n+2 = f(n)$. If both are negative, then $fm) = -4m > -4n = f(n)$. If m is negative and n is positive, then $f(m)$ is divisible by 4 and $f(n)$ is not. To see that f is onto, let $b \in A$. Then either $b \equiv 0(\bmod 4)$ or $b \equiv 2(\bmod 4)$. If $b \equiv 0(\bmod 4)$, then $f(-b/4) = -4(-b/4) = b$ and if $b \equiv 2(\bmod 4)$, then $f(\frac{b-2}{4}) = 4(\frac{b-2}{4}) + 2 = b$, so f is onto. This completes the proof.

10. (10 points) Solve: $a_n = -7a_{n-1} - 10a_{n-2}$ for $n \geq 2$ with the initial values $a_0 = 3$, $a_1 = 3$.

    Solution: Using $a_n = r^n$ yields the characteristic equation $r^2 + 7r + 10 = 0$, or $(r + 5)(r + 2) = 0$. Therefore the general solution is

$$a_n = c(-5)^n + d(-2)^n.$$

    The initial conditions give the system of equations

$$c + d = 3 \text{ and } -5c - 2d = 3.$$

    The solution to the system is $c = -3$ and $d = 6$. Hence, the solution to the recurrence relation is

$$a_n = (-3)(-5)^n + 6(-2)^n.$$

11. (20 points) Suppose $\mathcal{U}$ is a 12 element set with subsets A, B, and C satisfying $|A| = |B| = |C| = 6$, $|AB| = |BC| = 3$, $|AC| = 2$, and $|ABC| = 1$. Compute the following cardinalities.

(a) $|A \cup (B \cap \overline{C})|$

Solution: Use inclusion-exclusion to write $|A \cup (B \cap \overline{C})| = |A| + |(B \cap \overline{C})| - |A \cap (B \cap \overline{C})| = 6 + 3 - 2 = 7$

(b) $|(A \times A) \cup (B \times B) \cup (C \times C)|$

Solution: Use inclusion-exclusion to write $(|A\times A)\cup(B\times B)\cup(C\times C)| = |A\times A|+|B\times B|+|C\times C|-|(A\times A)\cap(B\times B)|-|(A\times A)\cap(C\times C)|-|(B\times B)\cap(C\times C)|+|(A\times A)\cap(B\times B)\cap(C\times C)| = 6^2+6^2+6^2-3^3-2^2-3^2+1 = 87$

(c) $|(A \cap B) \times (B \cap C) \times (A \cap C)|$

Solution: Notice that $|(A\cap B) \times (B\cap C) \times (A\cap C)| = |(A\cap B)| \times |(B\cap C)| \times |(A\cap C)| = 3 \cdot 3 \cdot 2 = 18$

(d) $|\overline{(A \cup B \cup C) \times (A \cup B \cup C)}|$

Solution: Its just $12 \cdot 12 - 11 \cdot 11 = 144 - 121 = 23$

(e) $|\overline{(A \cup B \cup C)} \times \overline{(A \cup B \cup C)}|$

Solution: Since $|\overline{(A \cup B \cup C)}| = 1$, it follows that $|\overline{(A \cup B \cup C)} \times \overline{(A \cup B \cup C)}|$ is also 1.