Math 1165 **Test 2**

November 13, 2002 Your name ____________________

There are a total of 150 points available on this test.

1. (20 points) Let $S = \{1, 2, 3, 4, 5, 6, 7, 8\}$. For each property below find the number of subsets T of S that have the property.

   (a) T has exactly three elements.

   Solution: This is just the number of ways to pick three things from an eight-element set, $C^8_3 = 56$.

   (b) $|T| = 5$ and T has three odd members and two even members.

   Solution: There are $C^4_1 = 4$ ways to choose the three odd members and $C^4_2 = 6$ ways to choose the two even members, so there are $4 \cdot 6 = 24$ such sets.

   (c) T has no prime number members. Recall that 1 is not prime.

   Solution: Since T must be a subset of $\{1, 4, 6, 8\}$, there are $2^4 = 16$ such subsets.

   (d) T has at least two odd and at least three even members.

   Solution: There are 11 ways to pick the odd members $(6 + 4 + 1)$ and 5 ways to pick the even members, so there are $11 \cdot 5 = 55$ such sets.

2. (20 points) Again let $S = \{1, 2, 3, 4, 5, 6, 7, 8\}$. For each property below find the number of five digit numbers that can be constructed using the members of S as digits and that have the property.

   (a) The number has a value of at least 20000.

   Solution: There are $E_5^8 = 8^5 = 32768$ five digit numbers exactly seven-eighths of which are larger than 20000. So there are $7/8 \cdot 32768 = 28672$ such numbers.

   (b) The number has a value of at least 20000 and the digits are all different.

   Solution: There are $P_5^8 = 6720$ five digit numbers exactly seven-eighths of which are larger than 20000. So there are $7/8 \cdot 6720 = 5880$ such numbers.

   (c) The number is a multiple of nine and the digits are all different.

   Solution: The sum of the digits of the number must be a multiple of 9, and the order of the digits is irrelevant. There are three five-subsets of S having an element sum of 18 and three with sum of elements 27. Since there are 5! permutations of each of these six sets. there are $6 \cdot 5! = 720$ such numbers.

   (d) The digits of the number are in increasing order from left to right. For example, 13457.

   Solution: Since every five-element subset of S gives rise to exactly one such number, there are $C_5^8 = 56$ such numbers.

3. (20 points) Let Z denote the set of all integers, $Z = \{0, \pm 1, \pm 2, \ldots, \}$, and let Z^+ denote the set of positive integers.

   (a) Find a <u>one-to-one</u> function g from Z^+ <u>onto</u> Z.

   Solution: One example that works is $$g(n) = \begin{cases} \frac{n-1}{2} & \text{if } n \text{ is odd} \\ -\frac{n}{2} & \text{if } n \text{ is even} \end{cases}$$

   (b) Prove that your function is one-to-one.

   Solution: Suppose x and y are different positive integers. We distinguish 3 cases. Case a. both x and y are odd. In this case $\frac{x-1}{2} \neq \frac{y-1}{2}$. Case b. both x and y are even. In this case $-\frac{x}{2} \neq -\frac{y}{2}$. Case c. One of them is odd and the other even. In this case, $g(x)$ and $g(y)$ have different signs.

   (c) Prove that your function is onto.

   Solution: If $n \geq 0$ then $g(2n+1) = \frac{2n+1-1}{2} = n$. In case $n < 0$, then $g(-2n) = -\frac{-2n}{2} = n$.

4. (20 points) Let A and B be sets with characteristic functions f_A and f_B. Compute the following characteristic functions of the terms of f_A and f_B.

   (a) $f_{A\cap B}$
   Solution: $f_{A\cap B} = f_A \cdot f_B$

   (b) $f_{A\cup B}$
   Solution: $f_{A\cup B} = f_A + f_B - f_A f_B$

   (c) $f_{\overline{A}}$
   Solution: $f_{\overline{A}} = 1 - f_A$

   (d) Use the properties (a),(b), and (c) to compute the characteristic function of $A\cup(B\cup C)$. How does this anticipate the inclusion-exclusion principle for three sets?

   Solution:

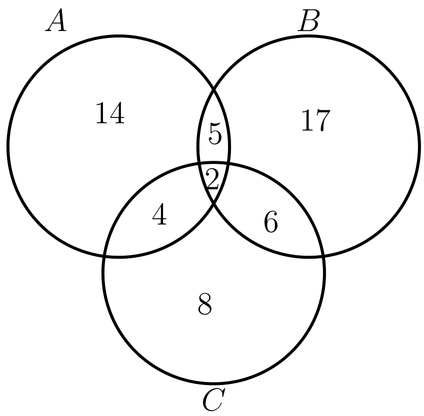
$$\begin{aligned} f_{A\cup(B\cup C)} &= f_A + f_{B\cup C} - f_A \cdot f_{B\cup C} \\ &= f_A + f_B + f_C - f_B f_C - f_A(f_B + f_C - f_B f_C) \\ &= f_A + f_B + f_C - f_A f_B - f_A f_C - f_B f_C + f_A f_B f_C \end{aligned}$$

5. (10 points) Suppose $f(n) = 5f(\frac{n}{2}) + n + 1$ and $f(4) = 40$. Find $f(1)$.

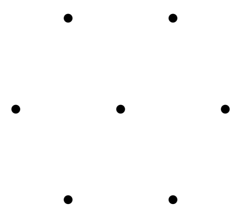
   Solution: First use $f(4)$ to find $f(2)$: $f(4) = 5f(\frac{4}{2}) + 4 + 1$. Therefore $40 = 5f(2) + 5$, or $f(2) = 7$. Then use $f(2)$ to find $f(1)$: $f(2) = 5f(\frac{2}{2}) + 2 + 1$. Therefore $7 = 5f(1) + 3$, or $f(1) = \frac{4}{5}$.

6. (15 points) Suppose one hundred students are polled about the academic preferences. Let A, B, and C denote the student who enjoy studying anthropology, biology, and chemistry respectively. For convenience, write AB to mean $A \cap B$, etc. Recall that $|X|$ denotes the number of elements of the set X. You are given the following information about the hundred students: $|ABC| = 2, |AB| = 7, |A \cup C| = 39, |\overline{A}BC| = 6, |A| = 25, |B| = 30,$ and $|C| = 20$. How many students do not enjoy any of the three types of courses?

   Solution: First note that $|AC| = |A| + |C| - |A \cup C| = 25 + 20 - 39 = 6$. Then $|A \cup B \cup C| = |A| + |B| + |C| - |AB| - |BC| - |AC| + |ABC| = 25 + 30 + 20 - 7 - 8 - 6 + 2 = 56$. So there are $100 - 56 = 44$ elements in $\overline{A \cup B \cup C}$.



7. (10 points) The points shown are the vertices of a regular hexagon with side length 1 together with the center of the hexagon. How many circles of radius 1 in the same plane have at least two vertices in the set?



Solution: 13. Each pair of adjacent points determines two circles. Counting the centers of such circles, we find that the seven points given all work. Also, there are six more centers outside the hexagon.

8. (15 points) Find the number of integers from 1 to 600 inclusive that are not divisible by any of the numbers $3, 5$, and 7

Solution: Let A be the integers from 1 to 600 divisible by 3 and B be the set of integers from 1 to 600 divisible by 5, and C the set of such integers not divisible by 7. The number of positive integers less than or equal to 600 that are divisible by none of 3, 5, and 7 is $|A \cup B \cup C| = |A| + |B| + |C| - |AB| - |BC| - |AC| + |ABC| = \lfloor \frac{600}{3} \rfloor + \lfloor \frac{600}{5} \rfloor + \lfloor \frac{600}{7} \rfloor - \lfloor \frac{600}{15} \rfloor - \lfloor \frac{600}{21} \rfloor - \lfloor \frac{600}{35} \rfloor + \lfloor \frac{600}{105} \rfloor = 200 + 120 + 85 - 40 - 28 - 17 + 5 = 325$. So $600 - 325 = 275$ of the numbers are not divisible by any of $3, 5$, and 7.

9. (10 points) Solve: $a_n = 3a_{n-1} + 10a_{n-2}$ for $n \geq 2$ with the initial values $a_0 = 0$, $a_1 = 1$. Use your solution to find the value of a_{10}.

   Solution: Using $a_n = r^n$ yields the characteristic equation $r^2 - 3r - 10 = 0$, or $(r-5)(r+2) = 0$. Therefore the general solution is

   $$a_n = c(5)^n + d(-2)^n.$$

   The initial conditions give the system of equations

   $$c + d = 0 \text{ and } 5c - 2d = 1.$$

   The solution to the system is $c = 1/7$ and $d = -1/7$. Hence, the solution to the recurrence relation is

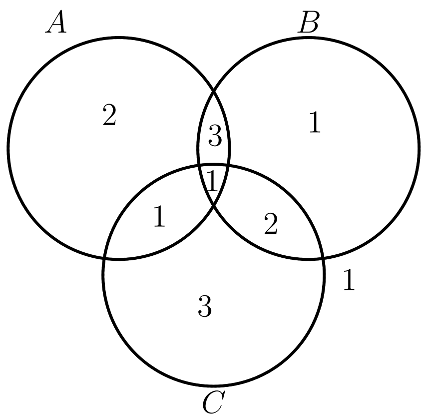
   $$a_n = (5^n - (-2)^n)/7.$$

   The value of a_{10} is 1394943.

10. (10 points) Suppose $\mathcal{U}$ is a 14 element set with subsets A, B, and C satisfying $|A| = |B| = |C| = 7$, $|AB| = 4$, $|BC| = 3$, $|AC| = 2$, and $|ABC| = 1$. Compute the following cardinalities.

    (a) $|A \cup (B \cap \overline{C})|$

    Solution: Use inclusion-exclusion to write $|A \cup (B \cap \overline{C})| = |A| + |(B \cap \overline{C})| - |A \cap (B \cap \overline{C})| = 7 + 4 - 3 = 8$. The Venn diagram for the sets is given by



    (b) $|(A \times A) \cup (B \times B) \cup (C \times C)|$

    Solution: Use inclusion-exclusion to write $(|A \times A) \cup (B \times B) \cup (C \times C)| = |A \times A| + |B \times B| + |C \times C| - |(A \times A) \cap (B \times B)| - |(A \times A) \cap (C \times C)| - |(B \times B) \cap (C \times C)| + |(A \times A) \cap (B \times B) \cap (C \times C)| = 7^2 + 7^2 + 7^2 - 4^2 - 3^2 - 2^2 + 1^2 = 119$

    (c) $|(A \cap B) \times (B \cap C) \times (A \cap C)|$

    Solution: Notice that $|(A \cap B) \times (B \cap C) \times (A \cap C)| = |(A \cap B)| \times |(B \cap C)| \times |(A \cap C)| = 4 \cdot 3 \cdot 2 = 24$

    (d) $|\overline{(A \cup B \cup C) \times (A \cup B \cup C)}|$

    Solution: Its just $14 \cdot 14 - 13 \cdot 13 = 27$

    (e) $|\overline{(A \cup B \cup C)} \times \overline{(A \cup B \cup C)}|$

    Solution: Since $|\overline{(A \cup B \cup C)}| = 1$, it follows that $|\overline{(A \cup B \cup C)} \times \overline{(A \cup B \cup C)}|$ is also 1.