Math 1165 **Test 2**

April 9, 2003 Your name ______________________

There are 138 points available on this test. You must show all your work.

1. (20 points) Let $\mathcal{U} = \{1,2,3,4,5,6,7,8,9,10\}$. Let $A = \{1,2,3,4\}, B = \{4,5,6,7\}$, and $C = \{6,7,8\}$. Recall that $\times$ denotes Cartesian product and $\overline{X}$ denotes the complement of X with respect to $\mathcal{U}$. Find each of the following. Recall that $|X|$ denotes the number of elements of the finite set X.

   (a) $|A \cup B \cup C|$

   (b) $|\overline{A} \times \overline{A}|$

   (c) $|(A \times B) \cup (B \times A)|$

   (d) $|\overline{A \cup B \cup C}|$

   (e) $|\,(A \times A) \cup (A \times B) \cup (A \times C)|$

2. (10 points) The digits $1, 2, 3, \cdots, 9$ are divided up into three groups, each with three elements. Prove that the product of the numbers in one of the groups must exceed 71.

3. (16 points) The Inclusion-Exclusion Principle.

   (a) Find the number of elements of $A_1 \cup A_2 \cup A_3$ if each of the three sets has cardinality 50, the intersection of any two of the sets has cardinality 30, and the intersection of all three sets has cardinality 10.

   (b) Find the number of elements in $A_1 \cup A_2 \cup A_3 \cup A_4$ if each set has cardinality 50, the intersection of any two sets has cardinality 30, each intersection of three sets has cardinality 10, and the intersection of all four sets has cardinality 2.

4. (10 points) You have a drawer full of socks. There are 4 red, 8 blue, 10 black, and 14 brown. How many socks must be removed from the drawer (in the dark) to be guaranteed that two of the socks removed match in color? Explain how the pigeonhole principle applies in this problem.

5. (20 points) Consider the recursively defined sequence $x_1 = 1, x_2 = 1$, and for $n > 2$, $x_n = \frac{x_{n-1}+1}{x_{n-2}}$.

   (a) Find the first 10 terms.

   (b) Suppose the first two terms are a and b. Find the next 5 terms in simplest form.

   (c) Prove that for any initial values, the sequence is periodic with period 5.

6. (15 points) Characteristic Functions. Recall that the characteristic function f_A of a set A is given by

$$f_A(x) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{otherwise} \end{cases}$$

   (a) If A and B are given sets with characteristic functions f_A and f_B, describe each of the following in terms of f_A and f_B.

      i. $f_{\overline{A}}$

      ii. $f_{A \cap B}$

      iii. $f_{A \cup B}$

   (b) Use characteristic functions to prove the DeMorgan Property below: that is, if A and B are any two sets, then $\overline{A} \cap \overline{B} = \overline{A \cup B}$.

7. (20 points) Let Z denote the set of all integers, $Z = \{0, \pm 1, \pm 2, \ldots, \}$, and let Z^+ denote the set of positive integers.

   (a) Find a <u>one-to-one</u> function g from Z^+ <u>onto</u> Z.

   (b) Prove that your function is one-to-one.

   (c) Prove that your function is onto.

8. (15 points) Solve: $a_n = 7a_{n-1} - 10a_{n-2}$ for $n \geq 2$ with the initial values $a_0 = 0$, $a_1 = 1$.

   (a) Find a_2, a_3, a_4, and a_5.

   (b) Find the general solution $a_n = c_1\lambda_1^n + c_2\lambda_2^n$.

   (c) Use the conditions $a_0 = 0$ and $a_1 = 1$ to find the unique solution to the recurrence relation.

   (d) Use part (c) to find a_{20}.

9. (12 points) Recall the function $f : [0,1] \to [0,1] \times [0,1]$ defined by

$$f(0.x_1x_2x_3\ldots) = (0.x_1x_3x_5\ldots, 0.x_2x_4x_6\ldots).$$

Compute each of the following. Leave your answer in the form in which the information is given (ie. fraction/fraction; decimal/decimal).

(a) $f(2/11)$

(b) $f(0.\overline{01})$

(c) Find x if $f(x) = (0, 1/9)$

(d) Find $f^{-1}(1/9, 1/9)$.