

April 9, 2003

Your name \_\_\_\_\_

There are 138 points available on this test. You must show all your work.

1. (20 points) Let  $\mathcal{U} = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$ . Let  $A = \{1, 2, 3, 4\}$ ,  $B = \{4, 5, 6, 7\}$ , and  $C = \{6, 7, 8\}$ . Recall that  $\times$  denotes Cartesian product and  $\bar{X}$  denotes the complement of  $X$  with respect to  $\mathcal{U}$ . Find each of the following. Recall that  $|X|$  denotes the number of elements of the finite set  $X$ .

(a)  $|A \cup B \cup C|$

**Solution:**  $A \cup B \cup C = \{1, 2, 3, 4, 5, 6, 7, 8\}$ , so  $|A \cup B \cup C| = 8$ .

(b)  $|\bar{A} \times \bar{A}|$

**Solution:**  $|\bar{A} \times \bar{A}| = 6 \times 6 = 36$ .

(c)  $|(A \times B) \cup (B \times A)|$

**Solution:**  $|(A \times B) \cup (B \times A)| = |(A \times B)| + |(B \times A)| - |(A \times B) \cap (B \times A)| = 16 + 16 - 1 = 31$ .

(d)  $|\overline{A \cup B \cup C}|$

**Solution:**  $|\overline{A \cup B \cup C}| = |\overline{\{1, 2, 3, 4, 5, 6, 7, 8\}}| = |\{9, 10\}| = 2$

(e)  $|(A \times A) \cup (A \times B) \cup (A \times C)|$

**Solution:**  $\{(A \times A) \cup (A \times B) \cup (A \times C)\} = \{(x, y) \mid x \in A \text{ and } y \in A \cup B \cup C\}$ , so  $|(A \times A) \cup (A \times B) \cup (A \times C)| = 4 \cdot 8 = 32$

2. (10 points) The digits  $1, 2, 3, \dots, 9$  are divided up into three groups, each with three elements. Prove that the product of the numbers in one of the groups must exceed 71.

**Solution:** The product of the 9 numbers is  $9!$ . Let  $a, b$  and  $c$  be the products of the members of each of the three groups. Then  $abc = 9!$ . But  $9!^{1/3} \approx 71.327$ . Suppose each of the numbers  $a, b$  and  $c$  are less than 71. Then  $abc < 71^3 < 9!$ , a contradiction.

3. (16 points) The Inclusion-Exclusion Principle.

- (a) Find the number of elements of  $A_1 \cup A_2 \cup A_3$  if each of the three sets has cardinality 50, the intersection of any two of the sets has cardinality 30, and the intersection of all three sets has cardinality 10.

**Solution:** By the inclusion-exclusion principle,  $|A_1 \cup A_2 \cup A_3| = 50 + 50 + 50 - 30 - 30 - 30 + 10 = 70$ .

- (b) Find the number of elements in  $A_1 \cup A_2 \cup A_3 \cup A_4$  if each set has cardinality 50, the intersection of any two sets has cardinality 30, each intersection of three sets has cardinality 10, and the intersection of all four sets has cardinality 2.

**Solution:** Using the Inclusion-Exclusion Principle for four sets

$$\begin{aligned} |A_1 \cup A_2 \cup A_3 \cup A_4| &= \\ \sum_{i=1}^4 |A_i| - \sum_{i \neq j} |A_i \cap A_j| + \sum_{i,j,k \text{ distinct}} |A_i \cap A_j \cap A_k| - |A_1 \cap A_2 \cap A_3 \cap A_4| &= \\ 4 \cdot 50 - \binom{4}{2} \cdot 30 + \binom{4}{3} \cdot 10 - 2 &= \\ = 200 - 180 + 40 - 2 = 58. \end{aligned}$$

4. (10 points) You have a drawer full of socks. There are 4 red, 8 blue, 10 black, and 14 brown. How many socks must be removed from the drawer (in the dark) to be guaranteed that two of the socks removed match in color? Explain how the pigeonhole principle applies in this problem.

**Solution:** Five socks is enough. Consider four pigeonholes, one for each color. Then five socks distributed among four pigeonholes is enough to guarantee that some pigeonhole contains at least 2 socks, and these two form a matching pair.

5. (20 points) Consider the recursively defined sequence  $x_1 = 1, x_2 = 1$ , and for  $n > 2$ ,  $x_n = \frac{x_{n-1}+1}{x_{n-2}}$ .

(a) Find the first 10 terms.

**Solution:** The first 10 terms are 1, 1, 2, 3, 2, 1, 1, 2, 3, 2.

(b) Suppose the first two terms are  $a$  and  $b$ . Find the next 5 terms in simplest form.

**Solution:** The first seven terms are  $a, b, \frac{b+1}{a}, \frac{a+b+1}{ab}, \frac{a+1}{b}, a, b$ . To get the term  $\frac{a+1}{b}$  requires skillful factoring.

(c) Prove that for any initial values, the sequence is periodic with period 5.

**Solution:** As above, let  $a$  and  $b$  denote the first two items. Then the third is  $\frac{b+1}{a}$  and the fourth is  $\frac{\frac{b+1}{a}+1}{b} = \frac{a+b+1}{ab}$ . Thus the fifth term is

$$\frac{\frac{a+b+1}{ab} + 1}{\frac{b+1}{a}} = \frac{ab + a + b + 1}{ab} \cdot \frac{a}{b+1} = \frac{a+1}{b}.$$

The sixth term is now easily found to be  $a$  and the seventh term is  $b$ . Thus, no matter what values we start with, the sequence is periodic.

6. (15 points) Characteristic Functions. Recall that the characteristic function  $f_A$  of a set  $A$  is given by

$$f_A(x) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{otherwise} \end{cases}$$

- (a) If  $A$  and  $B$  are given sets with characteristic functions  $f_A$  and  $f_B$ , describe each of the following in terms of  $f_A$  and  $f_B$ .

i.  $f_{\overline{A}}$

**Solution:**  $f_{\overline{A}} = 1 - f_A$

ii.  $f_{A \cap B}$

**Solution:**  $f_{A \cap B} = f_A \cdot f_B$

iii.  $f_{A \cup B}$

**Solution:**  $f_{A \cup B} = f_A + f_B - f_A \cdot f_B$

- (b) Use characteristic functions to prove the DeMorgan Property below: that is, if  $A$  and  $B$  are any two sets, then  $\overline{A \cap B} = \overline{A} \cup \overline{B}$ .

**Solution:** To prove that  $\overline{A \cap B} = \overline{A} \cup \overline{B}$ , it suffices to prove that  $f_{\overline{A \cap B}} = f_{\overline{A} \cup \overline{B}}$ . To that end, note that  $f_{\overline{A \cap B}} = f_{\overline{A}} \cdot f_{\overline{B}} = (1 - f_A)(1 - f_B) = 1 - f_A - f_B + f_A \cdot f_B$ . On the other hand,  $f_{\overline{A} \cup \overline{B}} = 1 - f_{A \cap B} = 1 - (f_A + f_B - f_A \cdot f_B)$ , which is just what we got above. Thus the characteristic functions are the same. This proves that the sets are equal.

7. (20 points) Let  $Z$  denote the set of all integers,  $Z = \{0, \pm 1, \pm 2, \dots\}$ , and let  $Z^+$  denote the set of positive integers.

(a) Find a one-to-one function  $g$  from  $Z^+$  onto  $Z$ .

**Solution:** One example that works is 
$$g(n) = \begin{cases} \frac{n+1}{2} & \text{if } n \text{ is odd} \\ -\frac{n}{2} & \text{if } n \text{ is even} \end{cases}$$

(b) Prove that your function is one-to-one.

**Solution:** Suppose  $x$  and  $y$  are different positive integers. We distinguish 3 cases. Case a. both  $x$  and  $y$  are odd. In this case  $\frac{x-1}{2} \neq \frac{y-1}{2}$ . Case b. both  $x$  and  $y$  are even. In this case  $-\frac{x}{2} \neq -\frac{y}{2}$ . Case c. One of them is odd and the other even. In this case,  $g(x)$  and  $g(y)$  have different signs and  $f(0) = 0$ .

(c) Prove that your function is onto.

**Solution:** If  $n \geq 0$  then  $g(2n + 1) = \frac{2n+1-1}{2} = n$ . In case  $n < 0$ , then  $g(-2n) = -\frac{-2n}{2} = n$ .

8. (15 points) Solve:  $a_n = 7a_{n-1} - 10a_{n-2}$  for  $n \geq 2$  with the initial values  $a_0 = 0$ ,  $a_1 = 1$ .

(a) Find  $a_2, a_3, a_4$ , and  $a_5$ .

**Solution:** The first few terms are  $a_2 = 7, a_3 = 39, a_4 = 203$ , and  $a_5 = 1031$ .

(b) Find the general solution  $a_n = c_1\lambda_1^n + c_2\lambda_2^n$ .

**Solution:** Using  $a_n = r^n$  yields the characteristic polynomial  $\lambda^2 - 7\lambda + 10$ . Factor it to find that the zeros are  $\lambda_1 = 2$  and  $\lambda_2 = 5$ , so the general solution is  $a_n = c_1\lambda_1^n + c_2\lambda_2^n$ .

(c) Use the conditions  $a_0 = 0$  and  $a_1 = 1$  to find the unique solution to the recurrence relation.

**Solution:** The two equations obtained are  $c_1 \cdot 2^0 + c_2 \cdot 5^0 = 0$  and  $c_1 \cdot 2 + c_2 \cdot 5 = 1$  which we can solve simultaneously to get  $c_1 = -1/3$  and  $c_2 = 1/3$ . This leads to the unique solution  $a_n = (5^n - 2^n)/3$ .

(d) Use part (c) to find  $a_{20}$ .

**Solution:** The value of  $a_{20}$  is  $(5^{20} - 2^{20})/3 = 3.17891435 \cdot 10^{13}$ .

9. (12 points) Recall the function  $f : [0, 1] \rightarrow [0, 1] \times [0, 1]$  defined by

$$f(0.x_1x_2x_3\dots) = (0.x_1x_3x_5\dots, 0.x_2x_4x_6\dots).$$

Compute each of the following. Leave your answer in the form in which the information is given (ie. fraction/fraction; decimal/decimal).

(a)  $f(2/11)$

**Solution:**  $f(2/11) = f(0.\overline{18}) = (0.\overline{1}, 0.\overline{8}) = (1/9, 8/9)$

(b)  $f(0.\overline{01})$

**Solution:**  $f(0.\overline{01}) = (0.\overline{0}, 0.\overline{1}) = (0, 1/9)$

(c) Find  $x$  if  $f(x) = (0, 1/9)$

**Solution:**  $(0, 1/9) = (0, 0.\overline{1})$ , so  $x = (0.\overline{01}) = 1/99$

(d) Find  $f^{-1}(1/9, 1/9)$ .

**Solution:** What this means is 'inverse of  $(1/9, 1/9)$ '. IE, what  $x$  satisfies  $f(x) = (1/9, 1/9)$ . It is not hard to see that this is  $1/9$ .