

## **Mathcounts**

### **Problems that combine algebra and geometry**

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#### **Problem One: The Mathematicians Banquet**

At the Mathematicians' Banquet, there was a cube-shaped cake that was frosted on all sides, and cut into smaller cube-shaped pieces.

Use the geometry of the cube and the algebra of patterns to solve the question: How many mathematicians were there at the banquet? You know that there were just enough pieces for everyone, and there were 8 times as many pieces with frosting on 0 sides as there were with frosting on 3 sides.

Hint: Think about cubes of different sizes: How many pieces will there be with frosting on 0, 1, 2, and 3 sides?

#### **Variations on the Theme: The Mathematicians' Banquet**

1. Sixty-four unit cubes were glued together to make a  $4 \times 4 \times 4$  cube. How many pairs of faces, one-by-one unit each, were glued together to make the larger cube?
2. A cube is painted red and then cut into 64 identical smaller cubes. How many of these cubes are painted red on at least one face?
3. A cube is painted red and then cut into 1000 identical smaller cubes. How many of these cubes are painted red on at least two faces?

4. A cube is made by gluing together 1000 identical smaller cubes. The cube is painted blue, and then taken apart. What percent of the small cubes are unpainted?
5. A large cube is made by gluing together identical smaller cubes. The cube is painted blue, and then taken apart. The number of unpainted cubes is the cube of the number of cubes painted on three sides. That is,  $(\text{number with three sides painted})^3 = (\text{number of unpainted})$ . How many small cubes were used to make the large cube?
6. **Cubes in Space:** Imagine a 10 by 10 by 10 cube made up of 1000 unit cubes and floating in space. What is the greatest number of unit cubes that can be seen by an observer at any moment in time?
7. A cube with edges measuring 10 cm is dipped into red paint. The cube is then divided into 125 smaller cubes. One cube is drawn, at random. What is the probability that the selected cube will have at least 25% of its surface area painted red?
8. A 5x5x5 box without a lid is completely filled with 125 cubes. How many of the cubes touch a side or the bottom of the box?
9. A wooden cube is painted red and then cut with six cuts into equal cubes. If the cubes are placed in a bag and one cube is drawn from the bag, what is the probability that it will have at most one side painted red?
10. Unit cubes are glued together to make a cube several units on each side. Some of the faces of this large cube are painted. When the cube is taken apart, there are exactly 45 cubes without any paint. How many faces of the large cube were painted?

### **Problem 2: Chuck's Problem**

A rectangular floor measuring 10 feet by 12 feet is tiled with one-foot square tiles. How many tiles would the diagonal of the rectangle cross?

### **Variations on the Theme: Chuck's Problem**

1. A rectangular floor is made from square tiles. The floor is 81 squares wide and 63 squares long. If a diagonal is drawn across the floor, how many squares will it cross?
2. The floor of a square room is covered in square tiles. If the diagonals of the room have a total of 21 tiles, how many tiles are on the floor?
3. A square tile floor has 29 square tiles along the two main diagonals. How many square tiles in the entire floor?

### Problem 3: Lattices

#### Lattice 1

Find the number that will appear directly below 25, above 100, below 98, below 100, ... when this lattice is continued in the same pattern.

				1					
				2	3	4			
			5	6	7	8	9		
	10	11	12	13	14	15	16		
17	18	19	20	21	22	23	24	25	
			.	.	.				

#### Lattice 2

Find the number that will appear directly below 25, above 100, ... when this lattice is continued in the same pattern.

				1				
				2	3			
				4	5	6		
				7	8	9	10	
				11	12	13	14	15

#### Lattice 3

In the lattice shown below, 19 is the 4<sup>th</sup> number in the 3<sup>rd</sup> row. If the lattice continues in the pattern shown, what will be the 16<sup>th</sup> number in the 2<sup>nd</sup> row?

				10				20		
			6	9			16	19		
		3	5	8		13	15	18	...	
	1	2	4	7	11	12	14	17	21	...

#### Lattice 4

In the lattice below, 11 is the fourth number in the second row. If the lattice continues in the same manner, what is the 20<sup>th</sup> number in the second row?

				6				12		18
		3	5		9	11		15	17	
	1	2	4	7	8	10	13	14	16	...

Lattice 5

If this spiral lattice continues in the same manner, what number will be directly above 100?...below 100?

	21	22	23	24	25	26
	20	7	8	9	10	27
	19	6	1	2	11	28
39	18	5	4	3	12	29
38	17	16	15	14	13	30
37	36	35	34	33	32	31

Lattice 6

The integers greater than 1 are arranged in 5 columns. If the pattern continues as shown below, which lettered column will contain the number 999?

A	B	C	D	E
	2	3	4	5
9	8	7	6	
	10	11	12	13
17	16	15	14	

Lattice 7

There are 9 tombstones arranged in a straight line to be counted forwards 1 to 9, and then backwards so stone number 8 will be named 10, and so on so that when you next reach number 1 it will be named 17. Which of the 9 original tombstones will be named 99?

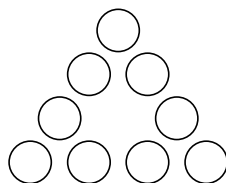
Lattice 8: A circular lattice

The numbers 1- $n$  are arranged in order around a circle. Find  $n$  if the number 15 is directly opposite from the number 40.

**Problem 4: Triangles**

The digits 1,2,3,..., 9 are arranged in the circles in the triangle shown so that the sum of the numbers on each side of the triangle is a constant.

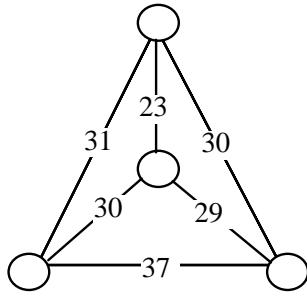
1. If the sum of the numbers on each side is 18, what is the sum of the numbers in the three corner circles?
2. If the sum of the numbers in the three corner circles is 12, what is the constant sum for the sides of the triangle?



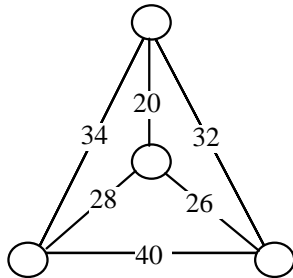
3. What are the smallest and largest sums possible?
4. Can you get all the sums between the smallest and largest?

The numbers on each line segment indicate the sum of the numbers in the two circles at the ends of the segment.

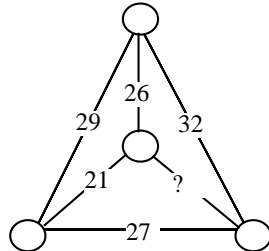
1. What is the smallest of the four numbers in the circles?



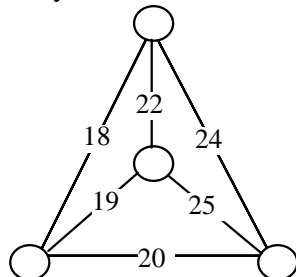
2. What is the sum of the four numbers in the circles?



3. In this diagram, exactly one of the numbers is missing. What is the missing number?

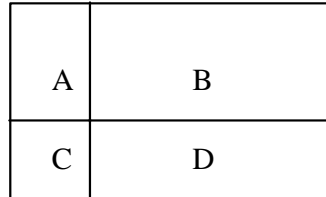


4. In this diagram, exactly one of the numbers is incorrect. What is the incorrect number?

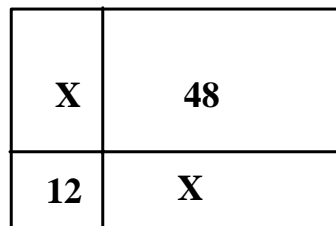


Problem 4: Rectangles

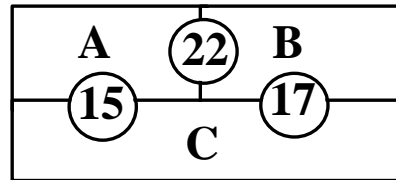
1. The diagram shows a rectangular garden divided into four smaller rectangles. Rectangle B has area 60 square meters, Rectangle C is a square with area 16 square meters, and Rectangle D has area 48 square meters. What is the number of meters in the perimeter of the garden?



2. The numbers and letters in each rectangular region of the rectangle represent the number of square inches in the area of the given region. The areas of the two regions labeled X are the same. Given that the dimensions of each of the regions are whole numbers, what is the number of inches in the difference between the greatest and least possible perimeters for the whole rectangle?



3. A square with area 100 square units is divided into a smaller square and four congruent rectangles. Given that all 5 pieces have equal areas, what is the number of units in the length of the longer side of one of the rectangles?
4. The regions A, B, and C in the rectangle contain numbers. The numbers in the circles represent the sums of the numbers in the regions touching each circle. What is the number in region C?



5. The areas of three faces of a rectangular prism with whole number dimensions are 54, 72, and 108 square in. What is the number of cubic in. in the volume of the rectangular prism?
6. The 9'' by 12'' rectangle ABCD is folded so that A and C coincide. How many inches are in the length of the crease?

### Problem 5: Probability

1. A bag contains 6 red marbles and 4 green marbles. I reach in the bag and draw out two marbles at the same time. What is the probability that they are both red?
2. There are two red balls and two white balls in a jar. One ball is drawn and replaced with a ball of the opposite color. The jar is shaken and one ball is chosen. What is the probability that this ball is red?
3. If two marbles are randomly removed without replacement from a bag containing blue and green marbles the probability they are both blue is  $\frac{1}{6}$ . If, instead, three marbles are randomly removed without replacement, the probability that all three are blue is  $\frac{1}{21}$ . What is the fewest number of marbles that must be in the bag?
4. There are 8 red (R), 8 blue (B), and 8 yellow (Y) marbles in a jar. What is the fewest number of marbles you can remove from the jar so that the ratio of the red to not red marbles remaining in the jar is 3 to 7, and the ratio of yellow to not yellow marbles remaining in the jar is also 3 to 7?
5. There are 12 red marbles and 12 black marbles in a jar. What is the least number of marbles that could be removed so that there are 4 red marbles for every 3 black marbles left in the jar?
6. The probability that a marble randomly chosen from a jar containing colored marbles is red or blue or yellow is  $\frac{1}{5}$ . A marble is drawn and returned to the jar. A second marble is drawn and returned to the jar. A third marble is drawn. What is the probability that none of the three marbles is red or blue or yellow?
7. There are fewer than 50 red, green or blue marbles in a bowl. The probability of drawing a red marble is  $\frac{2}{5}$  and the probability of drawing a green marble is  $\frac{3}{7}$ . Two marbles are drawn without replacement. What is the probability of drawing two blue marbles?
8. A bag contains 3 blue, 4 red and 3 yellow marbles. How many blue marbles must be added to the bag in order for the bag to contain 75% blue marbles?
9. There is a 50% chance of rain on Saturday and a 50% chance of rain on Sunday. What is the probability of 2 sunny days this weekend?
10. There are red and blue marbles in a jar. One third of the marbles are red. If one more red marble is put in the jar, then  $\frac{3}{8}$  of the marbles are red. How many blue marbles are in the jar?

### Problem 6: Proportion

1. My brother is 2 years old and I am 12 years old. In how many years will I be twice as old as my brother?
2. If Chen gives Elyse \$5, they will have the same amount of money. If, instead, Elyse gives Chen \$5, he will have three times as many dollars as she. How many dollars do they have together?
3. My brother is 2 years old and I am 12 years old. In how many years will I be twice as old as my brother?
4. If  $\frac{3}{4}$  of your money equals  $\frac{2}{3}$  of mine, what is the least whole number of dollars we could have all together?
5. In my class, one-half of the students went to the dance, one-third of the remainder went to the movies, and the remaining 8 people stayed home. How many students are in my class?

### Problem 7: Miscellaneous Problems

1. Find a five-digit number which when multiplied by 4 returns the same digits in reverse order.
2. I can get 3 records and 3 tapes for the same amount of money as I pay for 4 CD's. I have enough money to buy 28 CD's. How many of each can I buy, if I want an equal number of each and spend all my money?
3. What is the smallest integer that could be part of a set of two or more positive consecutive integers whose sum is 100?
4. A unit fraction is a fraction whose numerator equals one. The sum of three different positive unit fractions is  $\frac{7}{8}$ . Find the smallest number that could be the sum of the denominators of these fractions.
5. A unit fraction is a fraction whose numerator equals one. The sum of three different positive unit fractions is  $\frac{6}{7}$ . Find the smallest number that could be the sum of the denominators of these fractions.
6. The average of 12 **different** counting numbers is 12. What is the greatest possible value of one of these numbers?
7. A round table with radius 5 feet is pushed into a corner as shown. Point A is on the circumference of the table and is exactly 1 foot from one wall. How many feet is point A from the other wall?