

Tired Mascot

Each time our American football team scores (always 3 or 7 points), the mascot has to do as many pushups as his team has accumulated. So, for example, if the team first scored 3 points, he would do three pushups. Then, if they scored another 3 points, he would do 6 pushups. Finally, if the team scored 7 points, he would do 13 pushups for a total of $3+6+13 = 22$ pushups.

Before each game we buy a numbered sweepstake ticket and if the number equals the total number of pushups done by the mascot in the game, then we win a prize. This week my number can never win and so I've offered to swap it for any larger number because that would definitely increase my chance of winning. **What is the number on my ticket?**

Solution: The number is 566. The number of pushups the mascot must do is $7x + 3(T_n - x)$ where $T_n = 1 + 2 + \dots + n$, when the team scores n times during the game. Now the remainders when $T_n = n(n+1)/2$ is divided by 4 are 1, 3, 2, 2, 3, 1, 0, 0, and this sequence repeats. In other words, $T_1 \equiv 1 \pmod{4}$, etc. Let $K_n = \{7x + 3(T_n - x) \mid x = 0, 1, \dots, n\}$. Note that K_n is the number of possible pushups the mascot must complete when his team scores n times. For example $K_1 = \{3, 7\}$, $K_2 = \{9, 13, 17, 21\}$. It's easy to see that the elements of each K_n all differ by a multiple of 4 from one another. Thus, we have four problems, one for each remainder r .

For $r = 1$, we compute $K_1, K_6, K_9, K_{14}, K_{17}$, etc. Since $K_1 = \{3, 7\}$ and $K_6 = \{63, 67, \dots, 147\}$ and $K_9 = \{135, 139, \dots, 315\}$ and $K_{14} = \{315, 319, \dots\}$ the largest unachievable number is 59.

For $r = 2$, we must compute K_3, K_4, K_{11}, K_{12} and K_{19} . The largest member of K_{12} is $7 \cdot 12 \cdot 13/2 = 546$ and the smallest member of K_{19} is $3 \cdot 19 \cdot 20/2 = 570$, so 566 is the largest unachievable number of this sort.

For $r = 3$, we have $K_2 = \{9, 13, 17, 21\}$, $K_5 = \{45, 49, 53, \dots, 105\}$, $K_{10} = \{165, 169, \dots, 385\}$, and K_{13} has smallest member 273, we can see that 161 is the largest unachievable number of pushups for which the remainder is 2.

Finally, for $r = 0$, a similar analysis shows that the largest unachievable number of this sort is 116. Therefore, 566 is the largest unachievable number.