

# **My Favorite Problems**

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## My Favorite Problems

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### 1. Dinner Bill Splitting Problem.

In July, 1974, my neighbors agreed to celebrate with my wife and me our wedding anniversary. We went to a lovely restaurant, enjoyed a fine dinner, and asked for the bill. When it came, we asked that it be split as equally as possible. Realizing the waiter's discomfort, we all set to work on the problem. Soon someone at the table noticed that the bill was for an odd amount, so it could not be split perfectly. However, we realized that, except for the penny problem, we could take half the bill by simply reversing the dollars and the cents. In other words, if we double  $t$  dollars and  $s$  cents, the result differs by 1 cent from  $s$  dollars and  $t$  cents. We told the waiter about this. He was astounded: "I never knew you could do it that way." Several years later, over another dinner with mathematician friends, the question of uniqueness came up, and pretty soon we realized that this number is the only one with this surprising splitting property.

**Solution:** First, we'll state the problem in a more precise way. A dinner bill for  $s$  dollars and  $t$  cents is to be split as evenly as possible between two couples, where  $s$  and  $t$  are two-digit numbers. Since  $t$  is odd, the split cannot be exact. However, it turns out that twice  $t$  dollars and  $s$  cents differs by just one cent from  $s$  dollars and  $t$  cents. Find  $s$  and  $t$ . Symbolically, this means that  $|s.t - 2 \cdot t.s| = 0.01$  where  $s$  and  $t$  are two digit numbers. Let's convert this to an integer equation by multiplying by 100:

$$|100s + t - 2(100t + s)| = 1.$$

This is equivalent to  $|98s - 199t| = 1$  which can be interpreted as the two equations  $98s - 199t = 1$  or  $98s - 199t = -1$ .

Before continuing with the problem at hand, let's consider another problem whose solution will propel us forward solving the one at hand. The Decanting Problem is a liquid measuring problem that begins with two ungraduated decanters whose integer capacities  $a$  and  $b$  are given. The problem is to determine the smallest amount of liquid that can be measured and how such amount can be obtained, by a process of filling, pouring, and dumping. Specifically, there are three actions we can take:

- (a) fill an empty decanter,
- (b) dump out a full decanter, and
- (c) pour from one decanter to the other until either the receiving decanter is full or the poured decanter is empty.

Let's look at an easy one first. Let  $a = 3$  and  $b = 5$ . A little thought and we see that we can fill the 3 unit decanter twice, and dump the 5 unit decanter

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once to get 1 unit of liquid. Specifically,  $2 \cdot 3 - 1 \cdot 5 = 1$ . Next, suppose the decanters have capacities 5 units and 7 units. A little experimentation leads to the conclusion that 1 unit of water can be obtained by filling the 5 unit decanter 3 times, pouring repeatedly from the 5 unit to the 7 unit decanter and dumping out the 7 unit decanter twice. A finite state diagram is helpful to follow the procedure:

$$(0, 0) \implies (5, 0) \implies (0, 5) \implies (5, 5) \implies (3, 7) \implies (3, 0) \implies (0, 3) \implies (5, 3) \\ (5, 3) \implies (1, 7) \implies (1, 0),$$

where the notation  $(x, y)$  means the 5-unit container has  $x$  units of liquid and the 7-unit container has  $y$  units. Notice that the procedure includes 3 fills and 2 dumps, with fills and dumps alternating and separated by 4 pours. An arithmetic equation representing this is

$$3 \cdot 5 - 2 \cdot 7 = 1.$$

Notice that not only does the arithmetic equation follow from the state diagram, the reverse is also true. That is, given the arithmetic equation, it is an easy matter to construct the state diagram. In the next example, the least amount that can be measured is not 1. Let the decanters have sizes 15 and 99. Before reading on, can you see why it is impossible to obtain exactly one unit of water? An equation can be obtained for any sequence of moves. Such an equation is of the form

$$15x + 99y = z$$

where exactly one of the integers  $x$  and  $y$  is negative, and  $z$  is the amount obtained. Now notice that the left side is a multiple of 3, so the right side must be also. Thus the least amount that can be measured is 3 units. One can also reason this as follows: each fill adds a multiple of 3 units of water, each pour leaves the number unchanged, and each dump removes a multiple of three units, so the amount on hand at each stage is a multiple of 3.

In fact, the answer is that the least amount that can be measured is the greatest common divisor of the two decanter sizes, and the Euclidean algorithm tells us how to proceed. Suppose  $c = GCD(a, b)$ . The Euclidean algorithm yields a solution to

$$c = ax + by$$

where  $x$  and  $y$  are integers exactly one of which is positive and, except in trivial cases, the other is negative. For convenience, we assume  $x$  is positive. Then the solution to the decanting problem is to fill the  $a$  capacity decanter

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$x$  times, repeatedly pouring its contents into the  $b$  unit decanter. The  $b$  unit decanter will be dumped  $y$  times, so the total water on hand at the end is the difference  $ax - by = c$ .

Let's look at another specific example. Again we use the Euclidean Algorithm to solve the decanting problem. There are two stages. The first stage is a sequence of divisions. The second is a sequence of 'unwindings'. For this example, let the decanter sizes be  $a = 257$  and  $b = 341$ . Use the division algorithm to get  $341 = 1 \cdot 257 + 84$ . Then divide 257 by 84 to get  $q = 3$  and  $r = 5$ . That is  $257 = 3 \cdot 84 + 5$ . Continue dividing until the quotient  $q$  becomes 0. Thus 84 divided by 5 yields  $84 = 16 \cdot 5 + 4$ . Finally, divide 5 by 4 to get  $5 = 1 \cdot 4 + 1$ . This completes the first stage. Now to unwind, start with the final division scheme write  $1 = 5 - 1 \cdot 4$ . Then replace the 4 with  $84 - 16 \cdot 5$  to get  $1 = 5 - 1(84 - 16 \cdot 5)$ . This is equivalent to  $1 = 17 \cdot 5 - 1 \cdot 84$ . Check this to be sure. Then replace 5 with  $257 - 3 \cdot 84$  to get

$$1 = 17 \cdot (257 - 3 \cdot 84) - 1 \cdot 84,$$

i.e.,  $1 = 17 \cdot 257 - 52 \cdot 84$ . Finally, replace 84 with  $341 - 257$  to get  $1 = 17 \cdot 257 - 52(341 - 257)$ , which we can rewrite as

$$1 = 69 \cdot 257 - 52 \cdot 341.$$

Thus, the solution to the decanting problem is to measure out 1 unit of water by filling the 257 unit decanter 69 times, repeatedly pouring its contents into the 341 unit decanter, and, in the process, dumping out the 341 unit decanter 52 times.

Now back to the bill splitting problem. Imagining that we have two decanters with capacities  $a = 199$  and  $b = 98$ . Notice that  $GCD(199, 98) = 1$ . As we did above, we can use the Euclidean algorithm to find numbers  $x$  and  $y$  satisfying  $199x + 98y = 1$  where exactly one of the numbers  $x, y$  is negative. We do this by dividing repeatedly. First, 98 into 199 yields  $199 = 2 \cdot 98 + 3$ , Then 3 into 98 yields  $98 = 32 \cdot 3 + 2$  and finally we can write  $1 = 3 - 2$ . Next we go to the unwinding stage.

$$\begin{aligned} 1 &= 3 - 2 \\ &= 3 - (98 - 32 \cdot 3) \\ &= 3 - 98 + 32 \cdot 3 \\ &= 33 \cdot 3 - 1 \cdot 98 \\ &= 33(199 - 2 \cdot 98) - 98 \\ &= 33 \cdot 199 - 66 \cdot 98 - 98 \\ &= 33 \cdot 199 - 67 \cdot 98 \end{aligned}$$

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Thus, we have the values  $s = 67$  and  $t = 33$ . Indeed,  $2 \cdot 33.67 - 67.34 = -1$ .

2. The number  $N = 7 \cdot 5^{41}$  is a 30 digit number. Prove that some digit appears at least 4 times in the representation of  $N$ .

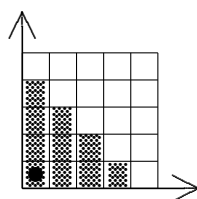
**Solution:** Suppose that every digit appears exactly 3 times. Then the sum of the digits is  $3(0 + 1 + 2 + \dots + 9) = 3 \cdot 45 = 135$  which is divisible by 9. This implies that  $N$  itself is divisible by 9. But the Fundamental Theorem of Arithmetic asserts that  $N$  has only one factorization into primes. The prime factors of  $N$  are 5 and 7, but not 3, so we have a contradiction.

3. A number is called 7-special if its decimal representation consists of only two digits, 0 and 7. For example,  $7/99 = 0.\overline{07}$  and  $7.707$  are such numbers. It is possible to write 1 as a sum of 7-special numbers. What is the fewest number of 7-special numbers whose sum is 1?

**Solution:** Divide 1 by 7 to get  $1/7 = .14285714\dots$ . If we can write  $1/7$  as a sum of number that require on the digits 0 and 1, then we can multiply all these summands by 7 to write 1 as a sum of 7-special numbers. Clearly we can write 1 as a sum of 8 numbers that require only the digits 0 and 1. Try  $a = .11111\dots$ ,  $b = .011110111110\dots$ ,  $c = .010111\dots$ ,  $d = .010111\dots$ ,  $e = .000111\dots$ ,  $f = .000101\dots$ ,  $g = .000101\dots$  and  $h = 000100\dots$

4. A Pebbling Problem. (M Kontsevich, 1981 Tournament of Towns).

The first quadrant is decomposed into squares for the following game. Some of these squares are occupied by counters. A position with counters may be transformed to another position according to the following rule: If the neighboring squares to the right and above a counter are both free, it is possible to remove the counter and replace it with counters at both these free squares. The goal is to have all the shaded squares free of counters. Is it possible to reach this goal if the initial position has just one counter in the lower left hand corner?



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**Solution:** Before we can solve this problem, we need some preliminary material. Consider the following problem. Find a pair of integers  $m$  and  $n$  such that

$$m/n = 0.3636363\dots$$

You've all seen problems like this in your algebra course, reinforcing the idea that every rational number has a repeating decimal representation. To solve it let  $x = 0.3636363\dots$  (which we can write as  $0.\overline{36}$ ) in which case  $100x = 36.3636363\dots$ . Subtract the former equation from the later to get  $99x = 36$ , which leads to  $m = 4$  and  $n = 11$ . This is really the same type of problem (in disguised form) as the following. Find the value of the geometric series

$$\frac{2}{3} + \frac{2}{9} + \frac{2}{27} + \dots$$




Again, give the answer a name. Let  $S = \frac{2}{3} + \frac{2}{9} + \frac{2}{27} + \dots$ . Then  $3S = 3 \cdot \frac{2}{3} + 3 \cdot \frac{2}{9} + 3 \cdot \frac{2}{27} + \dots = 2 + \frac{2}{3} + \frac{2}{9} + \frac{2}{27} + \dots = 2 + S$ . Subtract just as before to get  $2S = 2$  or  $S = 1$ . Try this with the geometric series  $.9 + .09 + .009 + \dots = 0.\overline{9}$  for a result that may surprise you. We can now solve the general problem: find  $S = a + ar + ar^2 + \dots$  where  $a$  and  $r$  are given and  $|r| < 1$ . Multiply both sides by  $r$  to get  $rS = ar + ar^2 + ar^3 + \dots$  and subtract to find that  $S - rS = a$ , in which case  $S = \frac{a}{1-r}$ . Now back to the pebbling problem. Assign each square a value  $v(i, j)$ ,  $i = 0, 1, 2, \dots, j = 0, 1, 2, \dots$  as follows:  $v(i, j) = 2^{-(i+j)}$ . Thus we have values as shown in the grid:

$\frac{1}{8}$					
$\frac{1}{4}$	$\frac{1}{8}$				
$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{8}$			
1	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{8}$		

Next let us define the value of a *position* of the puzzle. Each move, that is each replacement of a counter by two counters, results in a new position of the puzzle. The *value* of a position of the puzzle is the sum of the values of the squares

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









occupied in that position. The value of the initial position  is 1. We compute the value of various positions of the puzzle. The position  has value  $v = 1/2 + 1/2 = 1$ , while  has value  $v = 1/2 + 1/4 + 1/4 = 1$ . Is it clear that the value of each position is obtainable from the value of the previous position by removing  $1/2^{-n}$  from the sum and replacing it by  $1/2^{-n-1} + 1/2^{-n-1}$ , thus, not changing the value. Now, if there is a position of the puzzle where all the counters are outside of the shaded region, such a position must have the value 1. However, let us compute the sum of the values of all the squares outside the shaded region. We'll do this column by column. The values of the squares in the first column are  $1/16 + 1/32 + 1/64 + \dots = 1/8$ , so let us enter the number  $1/8$  for bookkeeping purposes.

$\frac{1}{8}$					











Notice next that second, third, fourth and fifth columns have the same sum,  $1/8$ . Now our diagram looks like

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$\frac{1}{8}$					
	$\frac{1}{8}$				
		$\frac{1}{8}$			
			$\frac{1}{8}$		
				$\frac{1}{8}$	

The sixth, seventh, etc columns have sum  $1/16, 1/32, 1/64$ , etc. Now our diagram looks like

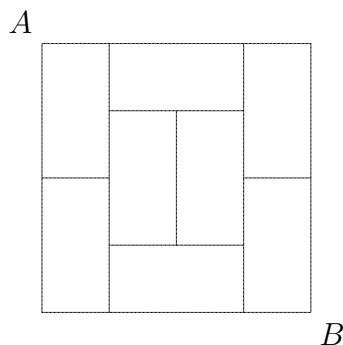
$\frac{1}{8}$							
	$\frac{1}{8}$						
		$\frac{1}{8}$					
			$\frac{1}{8}$				
				$\frac{1}{8}$	$\frac{1}{16}$	$\frac{1}{32}$	$\frac{1}{64}$

We can add the values of the bottom row in the usual way to get  $1/4$ . Hence the value of the entire first quadrant, minus the 10 shaded squares is  $1/8 + 1/8 + 1/8 + 1/8 + 1/4 = 3/4$ , so it is impossible to move all the counters outside the shaded area.

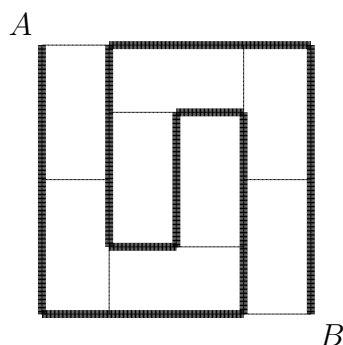
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5. Longest path problem. Each rectangle in the diagram is  $2 \times 1$ . What is the length of the longest path from  $A$  to  $B$  that does not retrace any part of itself? Prove that your answer is the best possible.



**Solution:** We think of the grid as a graph, with 18 vertices, and 25 edges. The vertices have *degrees* 2 and 3. There are four vertices of degree 2 and 14 of degree 3. The edges also come in two types, those with length 2 (there are 7 of these) and those with length 1 (there are 18 of these). Since we are starting at  $A$ , only one of the edges adjacent to  $A$  can be part of the path. The same is true for  $B$ . At all the other vertices, we can and must use exactly two edges. At the corners we have only two edges, but at all the other vertices, we must not use one of the incident edges. It is possible to choose a path so that the edges that are not used all have length 1. If every vertex belongs to the path, then exactly 17 edges can belong to the path, which means 8 edges do not belong. If all these edges have length 1, then the length of the path must be maximal,  $L = 2 \cdot 7 + 18 \cdot 1 - 8 \cdot 1 = 14 + 18 - 8 = 24$ . Note below that the path from  $A$  to  $B$  of length 24.



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6. The \$7.11 problem. A man goes into a convenience store, picks out four items, and goes to check out. The clerk tells him that her cash register is broken, and she will use her calculator. She proceeds to process the four amounts, and says, “that will be \$7.11”. “Wait a minute”, he protests, “you multiplied the prices together”. She promptly repeats the calculation, this time adding the four amounts, and exclaims, “there, you owe \$7.11, just as I said.” (There is no tax on food in this state.) There are two questions. First, what is the name of the convenience store, and what are the four prices? Challenge: try this problem with only three items. You’ll have to change the \$7.11, of course. Then try the problem for just two items. There are lots of solutions. Find them all. Then try the \$7.11 problem with three items and a total bill of \$8.25. Find some other total cost that could be used to solve the three item \$7.11 problem.

**Solution:** The four prices are \$1.25, \$1.20, \$1.50 and \$3.16. To see how to get these numbers, let  $\underline{x}$ ,  $\underline{y}$ ,  $\underline{u}$ , and  $\underline{v}$  denote the four prices, in dollars. Then  $\underline{xyuv} = 7.11$  and  $\underline{x+y+u+v} = 7.11$ . To eliminate the fractional part, multiply each of the unknowns and rename to get  $x = 100\underline{x}$ ,  $y = 100\underline{y}$ ,  $u = 100\underline{u}$ , and  $v = 100\underline{v}$ . Thus we have  $xyuv = 10^8 \cdot 7.11$  and  $x + y + u + v = 711$ . Factor the former to get  $xyuv = 711 \cdot 10^6 = 2^6 \cdot 3^2 \cdot 5^6 \cdot 79$ . It follows that exactly one of  $x, y, u, v$  must be a multiple of 79. For convenience, let’s say its  $v$ . Then  $v = 79, 158, 237$ , or  $316$ . We start by examining the last choice,  $316$ . In this case,  $xyu = 711 \cdot 10^6 \div 316 = 2^4 \cdot 3^2 \cdot 5^6$  and  $x + y + u = 711 - 316 = 395$ . Note that  $\sqrt[3]{2^4 3^2 5^6} = 50\sqrt[3]{18} > 125$ , so the sum  $x + y + u$  must be at least  $3 \cdot 125 = 375$ . Therefore we try to minimize  $x + y + u$  subject to  $xyu = 2^4 3^2 5^6$ . This occurs when we choose  $x, y$ , and  $u$  as close together as possible. Hence, let  $x = 5^3 = 125$ ,  $y = 2^3 \cdot 3 \cdot 5 = 120$  and  $u = 2 \cdot 3 \cdot 5^2 = 150$ . Thus the four prices are  $\underline{x} = \$1.25$ ,  $\underline{y} = \$1.20$ ,  $\underline{u} = \$1.50$  and  $\underline{v} = \$3.16$ . In the three item problem, we have the following: 6.00 (1, 2, 3); 8.25 (.75, 2, 5.5); 9.00 (.5, 4, 4.5); 10.80 (.4, 5, 5.4).

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7. You have three piles of stones containing 5, 49, and 51 stones. You can join any two piles together into one pile and you can divide a pile with an even number of stones into two piles of equal size. Can you ever achieve 105 piles each with one stone?

**Solution:** Note that both 5 and  $49 + 51$  are multiples of 5, so if the first move is to add the two big piles, then every pile after that will have a multiple of 5 stones. On the other hand, both  $5 + 49$  and 51 are multiples of 3, so with this first move, all the piles will have a multiple of 3 stones after that. Finally,  $5 + 51 = 56$  which, like 7 is a multiple of 7. No matter what moves are made after that, every pile in sight will have a multiple of 7 stones.

8. You sit at a table that has some coins on it. Each one, of course, is either showing heads or tails. You are wearing a blindfold and thick gloves, so it is impossible for you to tell by sight or touch what each coin is showing. At the outset, you know how many coins are on the table, and how many are showing heads. You can do whatever you like to the coins—turn them over, etc as long as all the coins end up back on the table. The question is how can you divide the coins into two groups so that each has the same number of heads?

**Solution:** Suppose there are  $k$  heads on the table. Take any  $k$  coins and make them into a group. Turn them all over. Suppose that among the  $k$  coins selected, there are  $h$  heads. Then the other group has  $k - h$  heads. Once all the coins in the group with  $k$  are turned over, there will be  $k - h$  heads in that group as well.