Available online at www.sciencedirect.com
SCIENCE DIRECT
JOURNAL OF Econometrics
Journal of Econometrics 119 (2004) 413–433
www.elsevier.com/locate/econbase

Stochastic volatility duration models

Eric Ghysels[a], Christian Gouriéroux[b,c,d], Joann Jasiak[e,*]

[a] University of North Carolina, USA
[b] CREST, France
[c] CEPREMAP, France
[d] University of Toronto, Canada
[e] Department of Economics, York University, 4700 Keele Street, Toronto, ON M3J 1P3, Canada

Abstract

We propose a class of two factor dynamic models for duration data and related risk analysis in finance and insurance. Empirical findings suggest that the conditional mean and (under) overdispersion of times elapsed between stock trades feature various patterns of temporal dependence. Therefore durations seem to be driven jointly by movements of two underlying factors. The paper presents a new model, called the stochastic volatility duration (SVD) model for processes that involve time varying uncertainty and time related risk. SVD-based estimation of market activity allows for the presence or absence of temporal interactions between the factors, depending on the market organization and the traded stock. The paper presents the distributional properties of SVD, and compares its performance to the performance of ACD models in an empirical study of intertrade durations of the Alcatel stock. Several new diagnostic tools for risk analysis are proposed, such as the conditional overdispersion and Time at Risk.
© 2003 Elsevier B.V. All rights reserved.

JEL classification: C41; D4

Keywords: Factor models; Duration; Liquidity risk; High frequency data; Time at risk

1. Introduction

The series of durations between events are encountered in many areas of science, engineering and economics. Among significant research topics are duration processes in environments which involve time varying uncertainty and time related risk. In financial econometrics, for example, the dynamics of times between trades (intertrade durations)

* Corresponding author.
E-mail addresses: eghysels@unc.edu (E. Ghysels), gouriero@ensae.fr (C. Gouriéroux), jasiakj@yorku.ca (J. Jasiak).

0304-4076/$ - see front matter © 2003 Elsevier B.V. All rights reserved.
doi:10.1016/S0304-4076(03)00202-1

reveal the current tendency of the stock market activity (liquidity). Likewise, temporal variation of times between car accidents provides to car insurance companies information on the behaviour of drivers. Since in such environments any change in the pattern of event arrivals may have implications in terms of costs, expected gains and risks, the ability of investors or insurance companies to make quick strategy adjustments is crucial. Consequently, the final outcome of their activities depends to a great extent on an adequate prediction of duration dynamics. The aim of this paper is to introduce a new class of dynamic models, called the stochastic volatility duration (SVD), that are suitable for the analysis of duration series and liquidity risk.

The common approach to dynamic duration modelling is based on the autoregressive conditional duration model (ACD) introduced by Engle and Russell (1998). The ACD model allows for both duration dependence and autoregressive dynamics of conditional expected durations.[1] The ACD model relies on the assumption that the ratios of durations Y_t and their conditional expectations Ψ_t are i.i.d. variables with a baseline distribution f_0, say:

$$Y_t/\Psi_t = u_t, \tag{1.1}$$

where u_t follows the f_0 distribution. The dynamics of conditional expectations is defined, for example, by the equation:

$$\Psi_t = \delta + \alpha\Psi_{t-1} + \beta Y_{t-1}, \tag{1.2}$$

in the ACD(1,1) model. This specification implies that the duration dynamics is captured exclusively by the conditional mean since all higher order conditional moments are still implicitly determined and tied down by the specification of conditional mean. This is a stringent assumption which in practice is often violated by the data and has restrictive implications for liquidity risk management in finance. Two approaches to liquidity risk analysis can be considered.

(i) First, we can examine liquidity risk by means of the two first conditional moments of intertrade durations. Under the ACD model, the conditional mean and variance of duration series are given by

$$\mathrm{E}(Y_t \mid \underline{Y_{t-1}}) = \Psi_t \quad \text{and} \quad V(Y_t \mid \underline{Y_{t-1}}) = k_0\Psi_t^2,$$

where the value k_0 depends on the baseline distribution. Even though overdispersion arises whenever $k_0 > 1$, its magnitude is supposed path independent. In contrast, empirical results based on intertrade durations on stock markets suggest the presence of path

[1] Other dynamic models for durations exist in the statistical literature. Usually, they assume an exponential model with path dependent intensity, but are less tractable than the ACD model. Zeger and Qaqish (1988) consider an exponential model, where the intensity parameter depends on lagged transformed durations. Smith and Miller (1986), Harvey (1989), Shephard (1994), Lunde (1997), Gamerman (1992) assume that the intensity parameter can be modelled as a stochastic variable whose conditional distribution is drawn from the conjugate family. Then, they derive explicit prediction and filtering formulas. Finally several authors have considered extensions of the ACD model based on a modification to Eq. (1.2). Typical examples are the fractionally integrated ACD model by Jasiak (1998) and the threshold model by Zhang et al. (2001).

dependent (under-)overdispersion as well as the existence of distinct dynamic patterns of the conditional mean and dispersion (see e.g. Section 4 for a discussion of Alcatel data and Giot (2000) for other examples of intertrade durations with conditional underdispersion).

We argue that at least two time varying factors are required to accommodate the complex duration dynamics. Intuitively, one factor may drive the conditional mean, while the second one the conditional (under-)overdispersion. Except for the Stochastic Conditional Duration model introduced by Bauwens and Veredas (1999), other models existing in the literature don't allow for duration dynamics of this complexity.

It is important to untie the dynamics of the conditional mean and dispersion (variance) of durations in order to obtain accurate results in the liquidity analysis of financial markets. The average length of intertrade durations reveals the speed of trading activity and is a natural indicator of market liquidity. On the other hand, the variance of durations represents the liquidity risk, i.e. the risk on time, and may be used as a choice criterion for asset allocation. Especially, the knowledge of intertrade durations is important to an investor who submits a limit order that remains queued in an electronic order book. Since the investor perfectly knows the future trading price, the uncertainty concerns only the time when the order is filled. The presence of two factors is also relevant when durations to trade a given volume are examined, or when they are price weighted (Gouriéroux et al., 1999; Engle and Lange, 1997). Since the weighted durations represent at least two structural latent variables, such as time and volume, capital or price, the corresponding models require at least two underlying factors.

(ii) We can also be concerned by the extreme liquidity risk. By analogy to the Value at Risk introduced for examining extreme risk on returns (see Gouriéroux and Jasiak (2001a, b) for a survey), we can define the Time at Risk (TaR) at level α:

$$P_t[Y_{t+1} > TaR_t(\alpha)] = \alpha, \tag{1.3}$$

where P_t denotes the conditional distribution at date t of the one step ahead duration Y_{t+1}.[2] $TaR(\alpha)$ defines the minimal time without a trade that may occur with probability α. The Time at Risk is a decreasing function of α and depends on the trade history. For an ACD model, we get:

$$\begin{aligned} P_t[Y_{t+1} > TaR_t(\alpha)] &= P_t\left[u_{t+1} > \frac{TaR_t(\alpha)}{\Psi_{t+1}}\right] \\ &= S_0\left(\frac{TaR_t(\alpha)}{\Psi_{t+1}}\right), \end{aligned}$$

[2] The Time at Risk can be extended to any horizon h by considering $TaR_t(\alpha, h)$ defined by

$$P_t[Y_{t+1} + \cdots + Y_{t+h} > hTaR_t(\alpha, h)] = \alpha.$$

Time at Risk can also be defined for durations weighted by volume or capital. Currently, it is unknown how to manage the liquidity risk and the risk on returns, that is how to use jointly the Value at Risk and the Time at Risk.

where S_0 is the survivor function associated with f_0. Thus:

$$TaR_t(\alpha) = \Psi_{t+1} S_0^{-1}(\alpha). \tag{1.4}$$

Due to the assumption of accelerated hazard, the Time at Risk functions $TaR_t(\cdot)$ evaluated from the ACD model are necessary parallel for all dates t.[3]

In this paper we try to improve upon the ACD model by considering a two-factor SVD model. The terminology of stochastic volatility (SV) is borrowed from the two factor discrete and continuous time models that represent the volatility of asset prices, and are used for evaluation of risk on returns, in particular for derivative security pricing (see for instance Ghysels et al. (1997) for a survey). Our motivation is similar since we intend to evaluate the risk on time. Obviously the specification proposed for the assessment of risk on time must differ from the stochastic volatility model for returns since the crude measures of time risk encountered in insurance or finance are overdispersion-based or TaR-based rather than variance-based. Our model extends the standard model of exponential durations with gamma heterogeneity. Two factors are introduced to allow for independent or interactive dynamics of the conditional mean and overdispersion that reproduce clustering and persistence.

The paper is organized as follows. In Section 2, we review the standard model of exponential durations featuring gamma heterogeneity and express it as a two factor model. Some straightforward transformations yield a Gaussian representation of the factors. In the next step we study various dynamic patterns of the Gaussian factors which drive the scale and overdispersion. In Section 3 we discuss several distributional properties of the SVD process and explain how they can be used for statistical inference. The paper follows with an empirical Section 4. We compare the SVD and ACD models estimated on intertrade duration data for the Alcatel stock listed on the Paris Bourse. For this purpose we introduce a set of diagnostic tools related to liquidity risk, such as the conditional Time at Risk and conditional overdispersion. Section 5 concludes.

2. Stochastic volatility duration models

2.1. The exponential model with gamma heterogeneity

The exponential duration model with gamma heterogeneity[4] is a standard approach developed in the duration literature (see e.g. Kalbfleisch and Prentice, 1980; Lancaster, 1990) for applications to cross-sectional and panel data. The SVD model extends this idea to time series of durations encountered, for example, in finance, where the number

[3] Note that Time at Risk is a conditional quantile. This function seems more suitable for liquidity risk analysis than the hazard function usually considered for duration data.

[4] The computational burden and complexity is greatly reduced by judicious choices of the mixing distribution. The gamma distribution is generally selected, since it is the conjugate distribution of the exponential family. When applied to count data and the Poisson family it leads to the Negative Binomial model (see e.g. Cameron and Trivedi, 2000) used in car insurance. Other mixing distributions have been sometimes considered such as the inverse gaussian distributions (Hougaard, 1984).

of consecutive observed durations can be very large. The initial model is based on the assumption that, conditional on the intensity λ, the duration variable Y is exponentially distributed with p.d.f.: $\lambda \exp(-\lambda y)$. Therefore the duration variable may be written as

$$Y = \frac{U}{\lambda}, \tag{2.1}$$

where U follows the exponential distribution with intensity one. Moreover, the intensity depends on a heterogeneity component V:

$$\lambda = aV, \tag{2.2}$$

where $V \sim \gamma(b,b)$ is independent of U and a and b are positive parameters.[5] The marginal distribution of this heterogeneity component is such that: $\mathrm{E}V = 1$ and $Var(V) = 1/b$. Thus, parameter a is equal to the expected intensity, whereas parameter b measures the magnitude of heterogeneity. The special case of homogeneity arises when V tends to one or alternatively when b tends to infinity. Finally the heterogeneity factor V can be integrated out to derive the marginal distribution of duration Y. This distribution is Pareto with p.d.f.: $h(y) = ab^{b+1}(ay+b)^{-b-1}$. The first and second order moments of Y exist if $b > 2$, and are given by: $\mathrm{E}Y = b/(a(b-1))$ and $VY = b^3/a^2(b-1)^2(b-2)$. It is well known that the presence of heterogeneity induces negative duration dependence, (see e.g. Lancaster, 1990) and overdispersion, since

$$\frac{VY}{(\mathrm{E}Y)^2} = \frac{b}{b-2} \geqslant 1.$$

Eq. (2.1) and (2.2) yield the exponential model with gamma heterogeneity:

$$Y = \frac{U}{aV}, \tag{2.3}$$

where U, V are independent, $U \sim \gamma(1,1)$ and $V \sim \gamma(b,b)$. This specification may be considered as a two factor model, where Y is a nonlinear function of U and V. Suitable nonlinear transformations can yield normally distributed factors. More explicitly, we get

$$Y = \frac{G(1,\Phi(F_1))}{aG(b,\Phi(F_2))} = \frac{H(1,F_1)}{aH(b,F_2)}, \tag{2.4}$$

where F_1, F_2 are i.i.d. standard normal variables, Φ is the c.d.f. of the standard normal distribution and $G(b,.)$ the quantile function of the $\gamma(b,b)$ distribution. We have: $H(1,F_1) = -\log[1-\Phi(F_1)]$. The function $H(b,F_2)$ has no simple analytical expression in the general case.

2.2. Extensions to dynamic models

The gaussian factor exponential duration model with gamma heterogeneity lends itself to dynamic generalizations. Indeed, we can introduce various dynamic patterns through the two underlying Gaussian factors. For instance, let us consider a bivariate

[5] Let us recall that the $\gamma(b,b)$ distribution admits the following p.d.f.:

$$f(y) = \frac{1}{\Gamma(b)} b^b y^{b-1} \exp(-by), \qquad \text{for } y \geqslant 0.$$

VAR time series representation of the process $F_t = (F_{1t}, F_{2t})'$, where the marginal distribution of F_t is constrained to be $\mathrm{N}(0, Id)$ to ensure that the marginal distribution of Y_t is Pareto.

This approach yields the class of SVD models, where

$$Y_t = \frac{1}{a}\frac{H(1,F_{1t})}{H(b,F_{2t})} \tag{2.5}$$

$$F_t = \sum_{j=1}^{p} \Psi_j F_{t-j} + \varepsilon_t, \tag{2.6}$$

Ψ_j, $j = 1,\ldots,p$ are 2×2 matrices, and ε_t is a Gaussian white noise with variance–covariance matrix $\Sigma(\Psi)$ such that $Var(F_t) = Id$. The observable process is simply a nonlinear transformation of an underlying bivariate gaussian process.[6] Since the Gaussian factor process is strictly stationary the duration process is strictly stationary too. Several important features can be captured by the factors in order to provide satisfactory fit in empirical work. Due to instantaneous correlations and autoregressive effects, these factors have generally joint effects on the conditional mean and overdispersion at any horizon.

To facilitate our understanding of the dynamic properties of the SVD model, let us consider the SVD of order $p = 1$. The model is defined by

$$Y_t = \frac{1}{a}\frac{H(1,F_{1t})}{H(b,F_{2t})}, \tag{2.7}$$

$$F_t = \Psi F_{t-1} + (Id - \Psi\Psi')\eta_t, \tag{2.8}$$

where

$$\Psi = \begin{pmatrix} \Psi_{11} & \Psi_{12} \\ \Psi_{21} & \Psi_{22} \end{pmatrix}$$

and (η_t) is a standard gaussian noise. As it is shown below, the model of order one can describe a large spectrum of empirical temporal dependence. One has to remember that the dynamics of the observed duration process will generally involve both nonlinearities and autocorrelation of large autoregressive order, due to the integration of an unobservable factor.

Let us now discuss some special patterns of temporal dependence.

(i) If $\Psi = 0$, we obtain i.i.d. durations: $Y_t = U_t/(aV_t)$, where the factors U_t and V_t are independent with time independent distributions: $U_t \sim \gamma(1,1)$ and $V_t \sim \gamma(b,b)$. The durations are Pareto distributed and feature overdispersion.

[6] The assumption of gaussian factors is not crucial. Indeed from the Volterra's expansion any strictly stationary process can be written as a nonlinear transformation of a gaussian process (even a white noise). The specificity of the model is that the transformation is static, that is does not involve lagged values of gaussian factors (see e.g. Shephard and Pitt, 1997, for similar assumption).

(ii) One of the factors may be held fixed in time. Let us consider the case when the matrix Ψ has all entries zero except for $\Psi_{22} = 1$; then the corresponding error term is equal to zero. We get $Y_t = (1/a)(U_t/V)$ with $U_t,\ t = 1, 2, \ldots, T$ and V being independent and $U_t \sim \gamma(1, 1)$, $V \sim \gamma(b, b)$. This case where the heterogeneity factor is time invariant is generally assumed in the car insurance and bonus-malus methodology to correct for moral hazard (see Gouriéroux et al., 1985; Dionne and Vanasse, 1989; Lemaire, 1995). Despite the unit root $\Psi_{22} = 1$, the duration process is still stationary. When Ψ_{22} tends to one, the corresponding error term ε_{2t} simultaneously tends to zero. However the duration process is not a regular stationary process. The autocorrelation function does not decay asymptotically to zero. It is constant at $1/(b-1)$ (see Appendix A), and increases with heterogeneity.

(iii) When Ψ is still a matrix with zero entries, except for Ψ_{22}, we model durations which are independent conditional on the heterogeneity component and allow for serial correlation of this component. Since the marginal distribution is Pareto, we have marginal overdispersion:

$$\frac{Var(Y_t)}{\mathrm{E}(Y_t)^2} = \frac{b}{b-2} \geqslant 1.$$

However, of greater interest are the (under-)overdispersions conditioned on the factors:

$$\frac{Var(Y_t \mid \underline{U_{t-1}}, \underline{V_{t-1}})}{\mathrm{E}(Y_1 \mid \underline{U_{t-1}}, \underline{V_{t-1}})^2} = \{2\mathrm{E}((1/V_t^2) \mid \underline{V_{t-1}}) - \mathrm{E}((1/V_t) \mid \underline{V_{t-1}})^2\} / \{\mathrm{E}((1/V_t) \mid \underline{V_{t-1}})\}^2$$

$$= \frac{2\,Var[(1/V_t) \mid \underline{V_{t-1}}]}{\mathrm{E}[(1/V_t) \mid \underline{V_{t-1}}]^2} - 1,$$

which can be path varying, larger or less than 1. Thus the second factor allows for a dynamic pattern of the conditional (under-)overdispersion. The same argument will apply to the observable (under-)overdispersion, conditioned on the lagged durations only. However the observable (under-)overdispersion is lower in average due to partial integration.

This particular model resembles an exponential ACD model: $Y_t = \Psi_t u_t$, where $u_t \sim \gamma(1, 1)$, $\Psi_t = \delta + \alpha\Psi_{t-1}$. Indeed, we get: $Y_t = \tilde{\Psi}_t u_t$, where $\tilde{\Psi}_t = [aH(b, F_{2,t})]^{-1}$ and $F_{2,t} = \Psi_{22}F_{2,t-1} + \sqrt{1 - \Psi_{22}^2}\varepsilon_{2,t}$. This special case of SVD model differs from the standard ACD-exponential model in two respects. First, the autoregression is written on $F_{2,t}$ which is a nonlinear transformation of $\tilde{\Psi}_t$ and not on $\tilde{\Psi}_t$ itself. This set-up ensures the positivity of $\tilde{\Psi}_t$. Second the autoregression involves an additional stochastic term to create the additional degree of freedom. [7]

(iv) When Ψ is zero except for Ψ_{11}, the durations are: $Y_t = U_t/(aV_t)$, where $V_1, \ldots, V_T$, $(U_1, \ldots, U_T)$ are independent and $V_t \sim \gamma(b, b)$. The conditional distribution of $Y_1, \ldots, Y_T$ given $U_1, \ldots, U_T$ is a product of inverse gamma distributions:

[7] Bauwens and Veredas (1999) proposed a logarithmic transformation of $\tilde{\Psi}_t$ and also introduced a stochastic term into the log $\tilde{\Psi}_t$ equation.

$1/Y_t \sim \gamma[bU_t/a]$ with p.d.f.:

$$f_t(y) = \frac{1}{\Gamma(b)} \exp\left[-\frac{bU_t}{ay}\right] \frac{1}{y} \left[\frac{bU_t}{ay}\right]^b.$$

(v) In general, when the Ψ matrix is not diagonal, we may generate effects of the lagged values of F_{1t} on F_{2t}, which are similar to the ARCH in the mean effect (see Engle et al., 1987). Indeed, a large or small lagged heterogeneity may have an impact on the future average duration level. Simultaneously we may also create strong autocorrelation in the second factor, which corresponds to the persistence in conditional overdispersion and may accommodate overdispersion clustering.

Let us now explain why we prefer to constrain the marginal distribution rather than the conditional distribution to be Pareto.

(i) A nonlinear dynamic model can be specified by either the conditional, or the marginal distribution. Even though the usual practice consists in selecting the form of the conditional distribution (see e.g. the usual ACD models), it is hard to find arguments in favour of this practice. Indeed while the marginal distribution is directly identifiable from the data, the conditional distribution has to be interpreted with respect to the information set that represents the history of the two unobservable factors. The point is that any modification to the ARMA structure of the factors also modifies the interpretation of the conditional distribution, whereas the interpretation of the marginal distribution remains unchanged.

(ii) Direct constraining of the marginal distribution has another advantage which is the possibility of separate estimation of the marginal distribution and of the model dynamics. This not only simplifies statistical inference, but also allows us to avoid misinterpretations. In contrast, when the model is specified by a fixed form of conditional distribution, the marginal distribution depends both on the model dynamics and the family of conditional distributions. When the model is misspecified the effects of misrepresenting any of these two elements are very difficult to disentangle.

(iii) Finally note another drawback of conditional Pareto distributions $P(a_t, b_t)$ (say), with dynamics imposed on the parameters a_t, b_t. For any type of stochastic parameter dynamics, the conditional duration distribution is Pareto and exhibits systematic conditional overdispersion. In contrast, the data may display conditional under- or overdispersion. The SVD model is compatible with this feature.

3. Distributional properties and statistical inference

In this section we make a brief review of the distributional properties of the SVD process and we discuss statistical inference.

(i) *Marginal distribution*: As noted before, the marginal distribution depends on the parameters a and b only. Thus the parameters can be estimated by maximizing the marginal likelihood function, without taking into account the temporal dependence. The estimators of the parameters a and b are:

$$(\tilde{a}_T, \tilde{b}_T) = Arg \max_{a,b} \sum_{t=1}^{T} \log h(y_t; a, b),$$

where $h(y;a,b) = ab^{b+1}/(ay+b)^{b+1}$. These estimators are consistent but not asymptotically efficient.

(ii) *Autocovariance function*: Let us assume a VAR(p) representation of factor dynamics [see (2.6)]. The duration Y_t is a static transformation of the factor $F_t = (F_{1,t}, F_{2,t})$: $Y_t = H(1, F_1, t)/aH(b, F_2, t)$ (say). The duration process $Y = \{Y_t\}$ is strongly stationary if the roots of the characteristic equation $\det(Id - \sum_{j=1}^{p} \Psi_j L^j) = 0$ lie outside the unit circle.

The expression of the autocovariance function of durations follows directly from the distribution of the pair $(F_{1,t}, F_{2,t})$. The autocovariances are given by

$$\begin{aligned} Cov(Y_t, Y_{t-h}) &= \frac{1}{a^2} Cov(\bar{H}(b, F_t), \bar{H}(b, F_{t-h})) \\ &= \frac{1}{a^2} \iint \bar{H}(b, f_t)\bar{H}(b, f_{t-h}) g[C_h; f_t, f_{t-h}] \, \mathrm{d}f_t \, \mathrm{d}f_{t-h} - \frac{1}{a^2} \frac{b^2}{(b-1)^2}, \end{aligned}$$

where $g[C_h; f_t, f_{t-h}]$ is the four dimensional normal p.d.f.:

$$\mathrm{N}\left(0, \begin{bmatrix} Id & C_h \\ C_h' & Id \end{bmatrix}\right),$$

with $C_h = Cov(F_t, F_{t-h})$. We deduce that the covariance may be written as

$$Cov(Y_t, Y_{t-h}) = \frac{1}{a^2} c[C_h; b] = \Gamma(h), \quad \text{(say)} \tag{3.1}$$

where c is a fixed function. Therefore the autocovariance function of the process (Y_t) is related to the autocovariance function of the factors (F_t) by a time independent continuous transformation. This indicates that the dynamics of (Y_t) inherits the features of the dynamics of (F_t). For instance, if the matrix C_h tends to zero for large h at an exponential decay rate ρ^h, the same property holds for $Cov(Y_t, Y_{t-h})$.[8]

The autocovariances can be used in a two-step method of simulated moments. In the first step the a and b parameters can be consistently estimated by the marginal maximum likelihood (see (i)). The autoregressive parameters $\Psi = (\Psi_j, \ j = 1, \dots, J)$ can next be derived from the relations:

$$\Gamma(h) = Cov(Y_t, Y_{t-h}) = \frac{1}{a^2} c[C_h(\Psi), b].$$

In the second step we may use the method of simulated moments after replacing the parameters a and b by their first step estimators $\tilde{a}_T, \tilde{b}_T$ (see McFadden, 1989; Pakes and Pollard, 1989; Duffie and Singleton, 1993; Gouriéroux and Monfort, 1996,

[8] When the heterogeneity is small, $b \approx \infty$, the computation of the four dimensional integral is simplified. The reason is that the function $H(b, F_2)$ admits the first order expansion $H(b, F2) \approx 1 + (1/\sqrt{b})F_2 \approx \exp(F_2/\sqrt{b})$, by the Central Limit Theorem. Therefore we have:

$$\mathrm{E}[\bar{H}(b, F_t)\bar{H}(b, F_{t-h})] \approx \mathrm{E}\left\{\log[1 - \Phi(F_{1t})]\log[1 - \Phi(F_{1,t-h})]\mathrm{E}\left[\frac{1}{\sqrt{b}}(F_{2t} + F_{2,t-h}) \,|\, F_{1t}, F_{1,t-h}\right]\right\}$$

and the conditional expectation $\mathrm{E}[\frac{1}{\sqrt{b}}(F_{2t} + F_{2,t-h}) \,|\, F_{1t}, F_{1,t-h}]$ may be computed analytically.

Chapter 2). The Ψ parameter is estimated by calibrating the empirical covariance $\hat{\Gamma}(h) = \frac{1}{T}\sum_{t=h+1}^{T}(Y_t - \bar{Y}_T)(Y_{t-h} - \bar{Y}_T)$ and the values of $\frac{1}{(\tilde{a}_T)^2}\hat{c}^s[C_h(\Psi), \tilde{b}_T]$, where $\hat{c}^s$ is a simulated approximation of c.

(iii) *The joint density function*: A limitation of nonlinear factor models is that the likelihood function is defined by a multidimensional integral, due to the presence of unobserved factors. To illustrate this point, let us derive the joint density function of Y_t's for a diagonal matrix of autoregressive coefficients:

$$\Psi_j = \begin{pmatrix} \psi_{1j} & 0 \\ 0 & \psi_{2j} \end{pmatrix},$$

$j = 1, \ldots, p$. Let us denote by $K(1,.)$ the inverse function of $H(1,.)$:

$$K(1,z) = \Phi^{-1}[1 - \exp(-Z)].$$

We get:

$$Y_t = \frac{1}{a}\frac{H(1, F_{1t})}{H(b, F_{2t})} \Leftrightarrow F_{1t} = K[1, aH(b, F_{2t})Y_t].$$

Let us now apply a change of variables in order to determine the conditional p.d.f. of $Y_1, \ldots, Y_T$ given $F_{21}, \ldots, F_{2T}$ and the initial condition, and then integrate out the second factor. The conditional p.d.f. is defined by the following multidimensional integral:

$$\begin{aligned} & l(y_1, \ldots, y_T \mid \underline{F}_{10}, \underline{F}_{20}, \underline{y}_0) \\ & = \int \cdots \int \frac{1}{(2\pi)^{\frac{T}{2}}} \frac{1}{(\sigma_1^2(\Psi_1))^{\frac{T}{2}}} \prod_{t=1}^{T} [aH(b, F_{2t})] \prod_{t=1}^{T} \frac{\mathrm{d}K}{\mathrm{d}Z}[1, aH(b, F_{2t})y_t] \\ & \times \exp\left\{ -\frac{1}{2\sigma_1(\Psi_1)^2} \sum_{t=1}^{T} (K[1, aH(b, F_{2t})y_t] - \sum_{j=1}^{p} \psi_{1j} K[1, aH(b, F_{2t})y_{t-j}])^2 \right\} \\ & \times \frac{1}{(2\pi)^{\frac{T}{2}}} \frac{1}{(\sigma_2^2(\Psi_2))^{\frac{T}{2}}} \exp\left[-\frac{1}{2\sigma_2^2(\Psi_2)^2} \sum_{t=1}^{T} \left(F_{2t} - \sum_{j=1}^{p} \psi_{2j} F_{2t-j} \right)^2 \right] \prod_{t=1}^{T} dF_{2t}. \end{aligned}$$

A standard approach consists in approximating this likelihood function by simulation and in applying a simulated maximum likelihood method (Shephard and Pitt (1997); Billio et al., 1998).

(iv) *The behaviour in a neighbourhood of the independence hypothesis*: Under the hypothesis of serial independence $\mathrm{H}_0 = (\Psi_1 = \cdots = \Psi_p = 0)$, the multidimensional integral simplifies to a product of unidimensional integrals, which may easily be computed (since the marginal distribution is a Pareto distribution). Simplifications arise also in a neighbourhood of the independence hypothesis. Indeed, when $\Psi_j \approx 0$, $j = 1, \ldots, p$, the (Y_t) process is locally a Markov process of order one in Ψ (Gouriéroux and Jasiak, 2001a):

$$l(y_t \mid \underline{y_{t-1}}; a, b, \Psi) = l(y_t \mid y_{t-1}; a, b, \Psi) + \mathrm{o}(\|\Psi\|^2).$$

It allows to approximate the p.d.f. by

$$l(y_1, \ldots, y_T; a, b, \Psi) \approx f(y_0; a, b) \prod_{t=1}^{T} l(y_t \mid y_{t-1}; a, b, \Psi),$$

which involves only unidimensional integrals. This approximate likelihood function can be used for two purposes. First, we can optimize the bounded memory log-likelihood:

$$(\tilde{a}, \tilde{b}, \tilde{\Psi}) = Arg \max_{a,b,\Psi} \sum_{t=1}^{T} \log l(y_t | y_{t-1}; a, b, \Psi).$$

This approach does not require computation of multidimensional integrals and provides consistent estimators (see Azzalini, 1983). Second, the first order expansion can be used to develop a Lagrange multiplier test of the independence hypothesis.

4. Empirical results

4.1. The Alcatel intertrade durations

We consider a sample of 5000 durations between trades of the Alcatel stock in July 1996 extracted from the records of the Paris Stock Exchange (SBF Paris Bourse). This stock belongs to the most heavily traded assets, and is included in the market index. In that period Paris market operated 5 days a week for 7 h daily. The original records contain data on all sell-buy contracts, including the prices, volumes and trading dates, with an accuracy of 1 s. The market operated under two regimes of matching procedures. At the opening at 10:00 a.m., the buy and sell orders arrived between 8:30 a.m. and 10:00 a.m. were aggregated, and an opening price was deduced from the equilibrium condition (call auction). During the day the buy and sell orders were matched as soon as possible (continuous auction). Since we are essentially interested in the intraday liquidity, the opening trades have been deleted from the sample. Among the intraday trades we may observe simultaneous trades. They occur when, for example, a large buy order is filled by several counterparts of the order book at different prices, in general. It is a common practice to aggregate simultaneous trades initialized by a single investor. After this data adjustment the file contains no zero durations. On average, the Alcatel stock is traded every 54.99 s and features a high intra-day variation with a standard deviation of 75.73 mainly due to a pronounced lunch time though in trades (Gouriéroux et al., 1999). The largest and the smallest durations in the sample are 1 and 1200 s (20 min), respectively. Usually duration data are characterized by strong periodicity. Table 1 presents the intra-day patterns of the first two unconditional sample moments.

The raw series is seasonally adjusted by following the method proposed by Engle and Russell (1998) under its multiplicative form (Bauwens and Giot, 2000). We regress

Table 1
Intraday seasonality: Hourly means and standard deviations, Alcatel

Hour	Mean	St. Dev.
10–11	46.42	52.97
11–12	46.39	51.01
12–13	77.55	100.27
13–14	125.79	156.51
14–15	58.03	76.22
15–16	48.48	56.83
16–17	39.64	50.79

the logarithm of the duration on the indicator variables that indicate the time of day. More precisely, the day is divided into K subperiods, and we consider the regression

$$\log y_t = \sum_{K=1}^{K} a_k x_{kt} + \varepsilon_t = a' x_t + \varepsilon_t,$$

where $x_{kt} = 1$, if date t belongs to the intraday subperiod k, and 0 otherwise. Then the seasonally adjusted series is defined by

$$\hat{y}_t = y_t \exp(-\hat{a}' x_t),$$

where $\hat{a}$ denotes the OLS estimator of a. Thus the duration is divided by a geometric average of the observed durations that correspond to the same time of day. The seasonally adjusted data have no measurement unit and have mean 1.03, standard deviation 1.28, and vary between 0 and 14.41. The dynamics of the raw and seasonally adjusted data are plotted in Figs. 1a and b, respectively.

The adjustment procedure has removed a wave-like pattern of the raw data. Both series of durations feature a tendency to cluster.

4.2. Estimation of the adjusted Alcatel series

We consider three dynamic specifications:

(i) a parametric ACD model

$$Y_t / \Psi_t = u_t,$$

where y_t follows an exponential distribution with intensity 1 and

$$\Psi_t = \delta + \sum_{j=1}^{p} \alpha_j \Psi_{t-j} + \sum_{k=1}^{q} \beta_k Y_{t-k}.$$

(ii) A semi-parametric ACD model, where the error term has an unconstrained distribution.

(iii) An SVD model of order 1 (see (2.5) and (2.6)).

The parameters δ, α, β of the ACD models have been estimated by the pseudo-maximum likelihood based on the exponential distribution. The p and q orders were

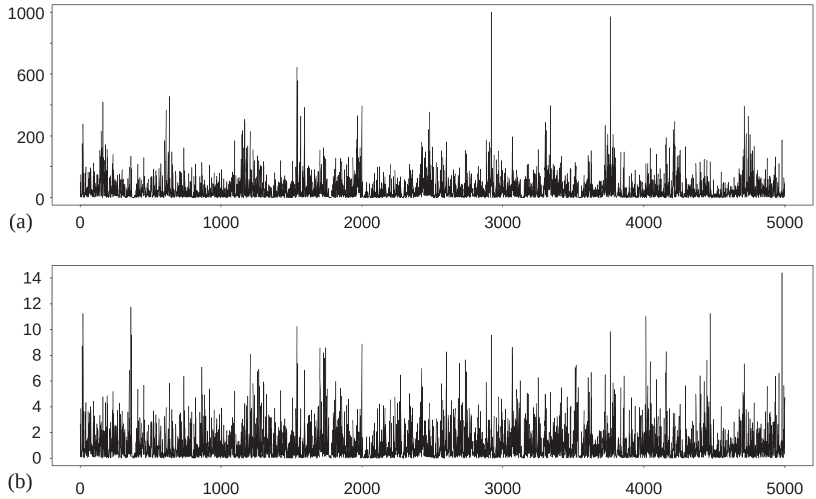

Fig. 1. (a) Alcatel durations. (b) Seasonally adjusted Alcatel durations.

selected by using the associated information criterion. The estimated orders are $\hat{p}=2$ and $\hat{q}=1$; the estimated dynamics of the conditional expected duration is

$$\Psi_t = \underset{(5.5)}{0.1709} + \underset{(3.5)}{0.5362\Psi_{t-1}} + \underset{(1.0)}{0.1361\Psi_{t-2}} + \underset{(7.3)}{0.1673Y_{t-1}},$$

where the numbers in parentheses represent the t-ratios.

For the semi-parametric ACD model, the unknown distribution of the errors ε_t is approximated by the empirical distribution of the residuals $\hat{\varepsilon}_t = Y_t/\hat{\Psi}_t$, where $\hat{\Psi}_t = \hat{\delta} + \hat{\alpha}_1\hat{\Psi}_{t-1} + \hat{\alpha}_2\hat{\Psi}_{t-2} + \hat{\beta}_1 Y_{t-1}$.

The SVD model has been estimated by the simulated method of moments. The selected moments include the marginal expectation and variance, selected autocorrelations of the duration series and of its squares. The estimation results are as follows:

Marginal density parameters:

$$\widehat{a/b} = 0.192 \ (3.4),$$

$$\hat{b} = 5.876 \ (6.2).$$

Autoregressive matrix:

$$\hat{\Psi} = \begin{pmatrix} 0.635 & 0.173 \\ 0.692 & 0.269 \end{pmatrix}.$$

Eigenvalues:

$$\hat{\lambda}_1 = 0.051\ (1.5),$$

$$\hat{\lambda}_2 = 0.991\ (7.1),$$

$$\hat{\mu}_1 = 0.060\ (1.7),$$

$$\hat{\mu}_2 = 0.843\ (6.9),$$

where λ_1, λ_2 (resp. μ_1, μ_2) are the eigenvalues of the matrix $\Psi\Psi'$ (resp. Ψ), and the numbers in parantheses are the t-ratios. Recall that μ_1, μ_2 provide the decay rates of the bivariate ACF of the factors, while λ_1, λ_2 are the canonical correlations.

4.3. Comparison of the ACD and SVD models

Let us first note that the SVD model involves six parameters a, b, Ψ. To get a fair comparison with the ACD model, we have retained the parametric ACD-exponential model with four parameters and a semi-nonparametric model, henceforth called simply ACD. The estimated models can be compared with respect to various criteria. Among the classical ones are the ability of reproducing the marginal distribution and the ACF computed from the data. For this purpose, simulated series of length 5000 were drawn for the ACD-exponential model, for the nonparametric ACD model (with errors drawn in the empirical distribution of the residuals) and for the SVD model (see Fig. 2). The marginal distributions were estimated by kernel smoothing of the empirical distribution.

The ACD-exp model and the SVD model fit well the marginal distribution of the duration data. This justifies ex-post the choice of the Pareto distribution as the marginal distribution of the durations for the Alcatel stock. When the marginal distribution features departures from a Pareto distribution, the SVD model is easily modified by changing the transformation which associates $F_{1,t}$ to U_t, that is by replacing the standard exponential distribution for U by another one, such as the Weibull distribution, for example. Typically the relation becomes:

$$Y = \frac{[H(1, F_1)]^c}{aH(b, F_2)},$$

where the power c is the additional positive parameter of the Weibull distribution. The ACD-exp and SVD models also provide good fit to the autocovariance function. Both models are successful in capturing the fast decrease of the empirical ACF for the first five significant lags. At lags higher than lag five, the autocorrelation function takes small values and is rather flat. This explains why a weak persistence effect is observed in the estimated ACD and SVD models. In particular, we get $\hat{\alpha}_1 + \hat{\alpha}_2 + \hat{\beta}_1 \approx 0.84$ for the ACD and a close value of μ_2 for the SVD (see Fig. 3).

Similarly, we can compare the ACF of squared durations, where both types of models show a poor fit (see Fig. 4).

These diagnostic tools have to be completed by additional criteria suitable for the conditional analysis of liquidity risk. These conditional criteria are computed for the simulated series, as well as for the data. We consider ten regimes of lagged durations:

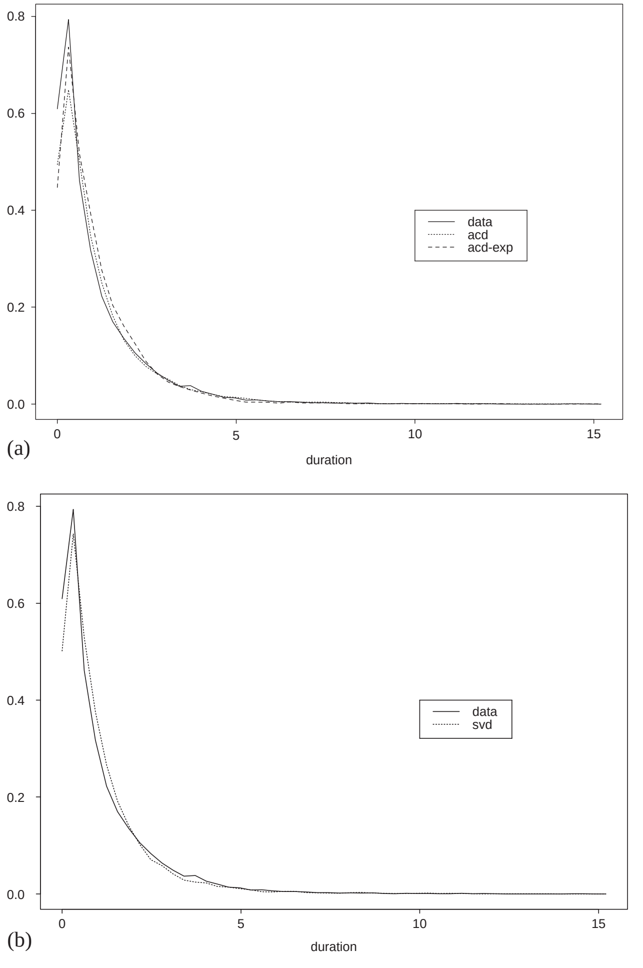

Fig. 2. Marginal density.

regime j occurs at time t, if Y_{t-1} lies between the $(j-1)$th and the jth deciles of the marginal distribution. Next, we compare various summary statistics conditioned on the regimes. These statistics are:

(i) the (conditional) Time at Risk associated with risk level α=10%, 15%, 20%, …, 85%;[9]

(ii) the (conditional) mean;

(iii) the (conditional) median and

(iv) the (conditional) overdispersion (see Figs. 5–8).

[9] Time at Risk is used as a diagnostic tool. In this application the computed Time at Risk cannot be used directly for liquidity risk control, since it is computed at a short horizon and from unweighted seasonally adjusted durations.

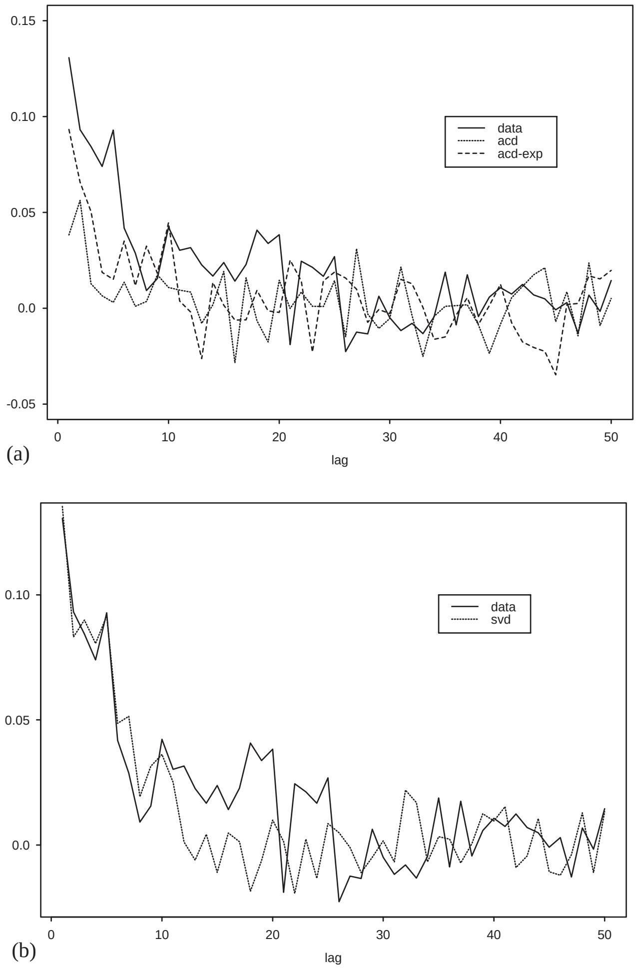

Fig. 3. Autocorrelation function.

Before discussing the figures, let us recall two properties of the ACD model, when the available information includes all lagged durations: the (conditional) overdispersion is constant, in particular equal to one for exponential errors and the Time at Risk for different levels α are parallel. These properties are not necessarily satisfied for the limited information set used in our comparative study. However we see in Fig. 8 that the conditional overdispersion is close to one for the ACD-exp model, takes larger values for the nonparametric ACD. None of these models is capable of reproducing the decrease in conditional overdispersion displayed by the data and the SVD simulation. The comparison in terms of the (conditional) Time at Risk is of greater importance (see Fig. 5). The TaR functions provide the information on the whole conditional distribution of Y_t. This quantile representation is more adequate for

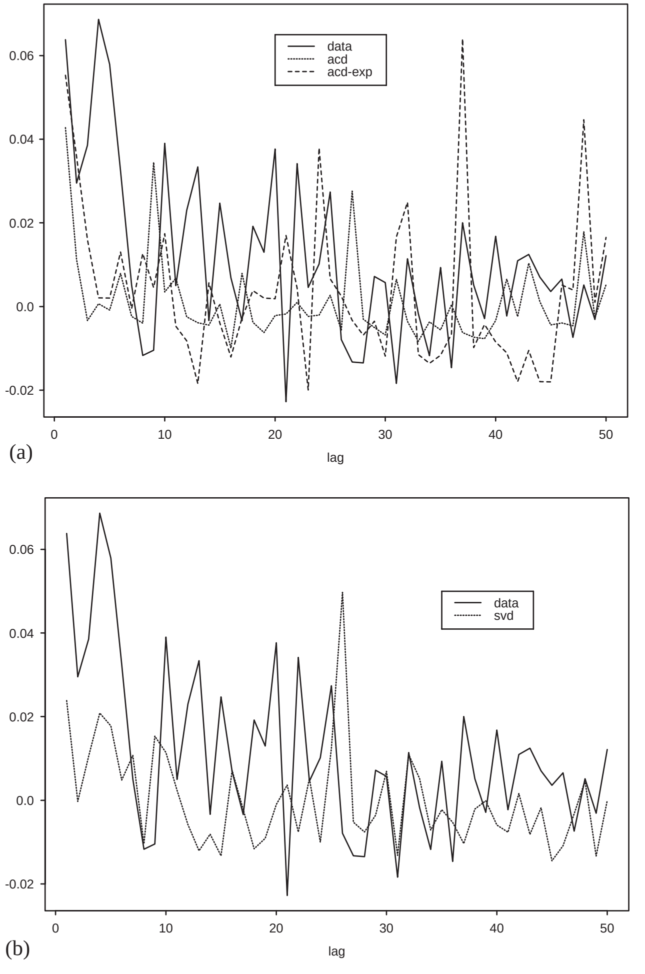

Fig. 4. Autocorrelations of squared durations.

interpretation in terms of liquidity risk then the standard density forecast, for example. Also in this case the ACD models have a difficulty to capture the increase in the Time at Risk for small risk levels (i.e. those that correspond to the largest values of Time at Risk). The same type of remarks can be made for the conditional mean and conditional median.

5. Conclusions

We proposed in this paper a class of two factor models for duration data. The approach is based on the standard exponential model with gamma heterogeneity and

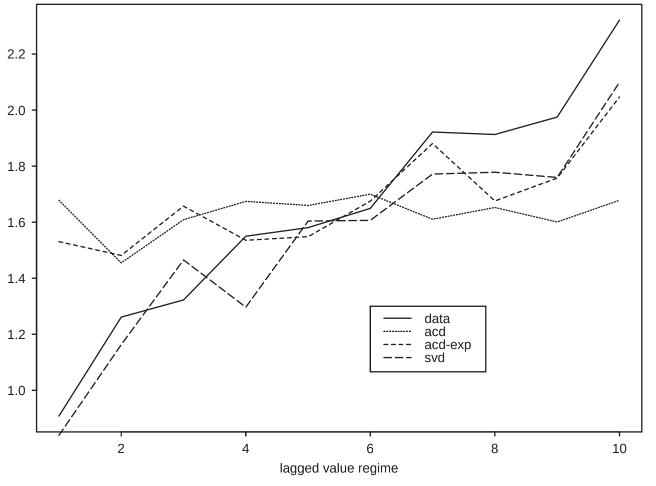

Fig. 5. Conditional Time at Risk.

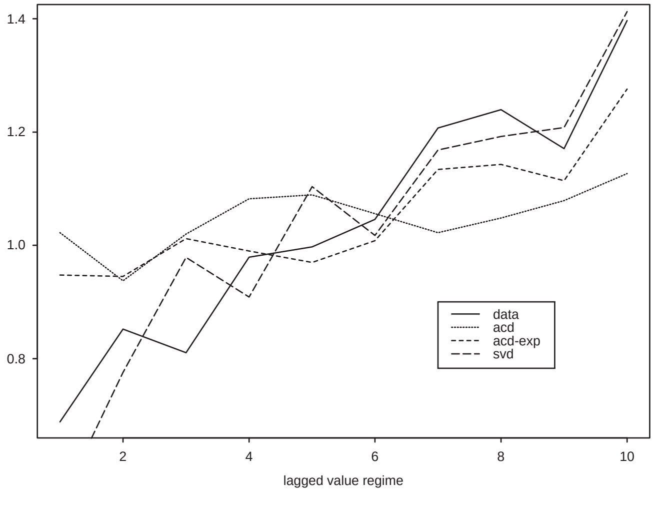

Fig. 6. Conditional mean.

its factor representation. The factors allow to introduce joint dynamics of the underlying conditional mean and (under-)overdispersion parameters for the analysis of liquidity on financial markets, or risk on car insurance contracts. The empirical performance of the new model was compared with the performance of different ACD models for intertrade durations. It seems that the introduction of an additional factor improves the analysis

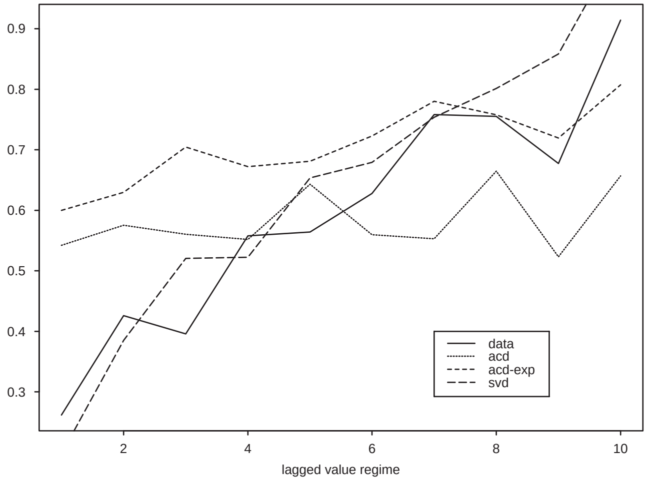

Fig. 7. Conditional median.

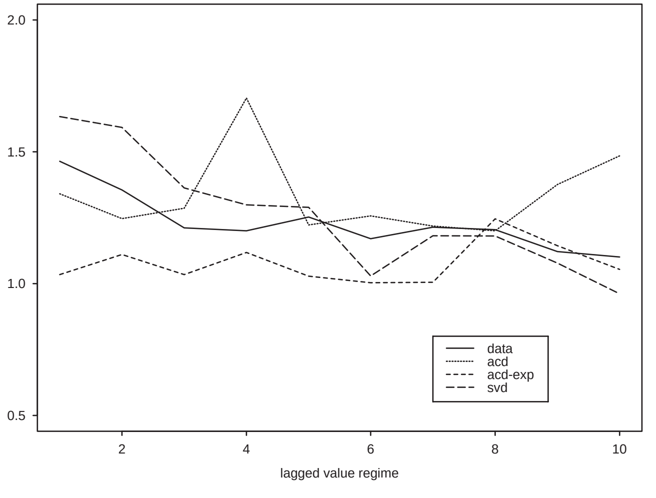

Fig. 8. Conditional overdispersion.

of extreme liquidity risk, as observed by comparing the conditional overdispersion and the conditional Time at Risk.

Acknowledgements

The third author gratefully acknowledges financial support of the Natural Sciences and Engineering Research Council of Canada.

Appendix A

A.1. First order correlation in the limiting case $\Psi_{22}=1$

The duration process is $Y_t = U_t/V$, where $U_1,\dots,U_t,\dots,V$ are independent. We get:

$$Cov(Y_t, Y_{t-1}) = Cov\left[\mathrm{E}\left(\frac{Y_t}{V}\right), \quad \mathrm{E}\left(\frac{Y_{t-1}}{V}\right)\right]$$

$$= Var\left(\frac{1}{V}\right),$$

$$Var(Y_t) = Var\left[\mathrm{E}\left(\frac{Y_t}{V}\right)\right] + \mathrm{E}\left[Var\left(\frac{Y_t}{V}\right)\right]$$

$$= Var\left[\frac{1}{V}\right] + \mathrm{E}\left[\left(\frac{1}{V}\right)^2\right].$$

Therefore the first order correlation is $1/(b-1)$. It takes positive values when b belongs to $(2,+\infty)$.

References

Azzalini, A., 1983. Maximum likelihood estimation of order m for stationary stochastic processes. Biometrika 70, 381–387.

Bauwens, L., Giot, P., 2000. The logarithmic ACD model: an application to the bid-ask quote process of three NYSE stocks. Annales d'Economie et de Statistique 60, 117–149.

Bauwens, L., Veredas, D., 1999. Stochastic Conditional Duration Model: a Latent Factor Model for the Analysis of Financial Durations. CORE Discussion Paper.

Billio, M., Monfort, A., Robert, C., 1998. The Simulated Likelihood Ratio Method. CREST D.P. 9821.

Cameron, C., Trivedi, P., 2000. Essentials of count data regression. In: Baltagi, B. (Ed.), Companion in Econometric Theory. Blackwell, Oxford.

Dionne, G., Vanasse, C., 1989. A generalization of actuarial automobile insurance rating models. The negative binomial distribution with a regression component. Actuarial Bulletin 19, 199–212.

Duffie, D., Singleton, K.T., 1993. Simulated moments estimation of Markov models of asset prices. Econometrica 61, 929–952.

Engle, R., Lange, J., 1997. Measuring, forecasting and explaining time varying liquidity in the stock market. Working Paper, University of California, San Diego.

Engle, R., Russell, J., 1998. The autoregressive conditional duration model. Econometrica 66, 1127–1162.

Engle, R., Lilien, D., Robbins, R., 1987. Estimating time varying risk premia in the term structure: the ARCH-M model. Econometrica 55, 391–407.

Gamerman, D., 1992. A dynamic approach to the statistical analysis of point processes. Biometrika 79, 39–50.

Ghysels, E., Harvey, A., Renault, E., 1997. Stochastic volatility. In: Maddala, G.S. (Ed.), Handbook of Econometrics, Vol. XIV, Statistical Methods for Finance, North-Holland, Amsterdam.

Giot, P., 2000. Time transformations, intra-day data and volatility models. Journal of Computational Finance 4, 31–62.

Gouriéroux, C., Jasiak, J., 2001a. Dynamic factor models. Econometric Reviews 20, 385–422.

Gouriéroux, C., Jasiak, J., 2001b. Value at risk. In: Ait-Sahalia, Y., Hansen, L.P. (Eds.), Handbook of Financial Econometrics, Elsevier, Amsterdam, forthcoming.

Gouriéroux, C., Monfort, A., 1996. Simulation Based Econometric Methods. Oxford University Press, Oxford.

Gouriéroux, C., Monfort, A., Trognon, A., 1985. Pseudo maximum likelihood methods: application to Poisson models. Econometrica 52, 701–721.

Gouriéroux, C., Jasiak, J., Le Fol, G., 1999. Intra-day market activity. Journal of Financial Markets 2, 193–226.

Harvey, A., 1989. Forecasting, Structural Time Series and the Kalman Filter. Cambridge University Press, Cambridge.

Hougaard, P., 1984. Life table methods for heterogeneous populations: distributions describing the heterogeneity. Biometrika 71, 75–83.

Jasiak, J., 1998. Persistence in intertrade durations. Finance 19, 166–195.

Kalbfleisch, J., Prentice, R., 1980. The Statistical Analysis of Failure Time Data. Wiley, New York.

Lancaster, T., 1990. The Econometric Analysis of Transition Data. Cambridge University Press, Cambridge.

Lemaire, J., 1995. Bonus Malus Systems in Automobile Insurance. Kluwer Academic Publishers, Dordrecht.

Lunde, A., 1997. A Conjugate gamma model for durations in transaction data. Working Paper, University of Aarhus.

McFadden, D., 1989. A method of simulated moments for estimation of discrete response models without numerical integration. Econometrica 57, 995–1026.

Pakes, A., Pollard, D., 1989. Simulation and the asymptotics of optimization estimation. Econometrica 57, 1027–1057.

Shephard, N., 1994. Partial non gaussian state space. Biometrika 81, 115–131.

Shephard, N., Pitt, M., 1997. Likelihood analysis of non gaussian measurement time series. Biometrika 84, 653–667.

Smith, S., Miller, J., 1986. A non gaussian state space model and application to prediction of records. Journal of the Royal Statistical Society B 48, 79–88.

Zeger, S., Qaqish, B., 1988. Markov regression models for time series. Biometrika 44, 1019–1031.

Zhang, M., Russell, J., Tsay, W., 2001. A nonlinear autoregressive conditional duration model with application to financial transaction data. Journal of Econometrics 104, 179–207.